



The structure of the wave operator in four dimensions in the presence of resonances

Angus Alexander¹ · Adam Rennie¹

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Abstract

We show that the wave operators for Schrödinger scattering in \mathbb{R}^4 have a particular form which depends on the existence of resonances. As a consequence of this form, we determine the contribution of resonances to the index of the wave operator.

Keywords Schrödinger operators · Resonances · Wave operator · Fredholm index

Mathematics Subject Classification 35P25 · 81U05

1 Introduction

In this paper we study scattering for Schrödinger operators in \mathbb{R}^4 . In particular, we analyse the structure of the wave operator in terms of the scattering operator, the generator of dilations, and a novel contribution corresponding to the existence of resonances. The form of the wave operator allows us to deduce the contribution of resonances to Levinson's theorem via the index of the wave operator.

Levinson's theorem [33] gives in dimension $n = 1$ that the number of eigenvalues (counted with multiplicity) of the Schrödinger operator $H = H_0 + V$ (with $H_0 = -\Delta$) for a suitably decaying potential V satisfies

$$N = \frac{1}{\pi} (\delta(0) - \delta(\infty)) + \frac{1}{2} \nu, \quad (1.1)$$

where δ is the scattering phase and $\nu \in \{0, 1\}$ depends on the existence of a resonance (a distributional solution to $(-\Delta + V)\psi = 0$ with $\psi \notin L^2(\mathbb{R})$).

In dimension $n = 3$, it was shown in [34, Section 5] that for spherically symmetric potentials, Eq. (1.1) holds for each angular momentum mode. This result was extended

✉ Angus Alexander
angusa@uow.edu.au

Adam Rennie
renniea@uow.edu.au

¹ School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, Australia

by [35, Section 7] for general potentials, and it was shown that resonances can occur and provide a half-integer contribution to a Levinson-type theorem.

One approach to resonances is the threshold behaviour of the spectrum of Schrödinger operators as the strength of the potential is scaled, $\lambda \rightarrow \lambda V$. A resonance occurs just as an eigenvalue emerges from the continuous spectrum. Klaus and Simon [31, Table I] determined in dimension $n = 2, 4$ that the logarithmic behaviour of the singularity of the free resolvent $R_0(z) = (H_0 - z)^{-1}$ near $z = 0$ gives resonances with different properties to those found in odd dimensions. In particular, in the case of a spherically symmetric potential the ‘resonances’ in dimension $n = 2$ are either s -resonances or p -resonances (corresponding to angular momentum $\ell = 0$ or $\ell = 1$, respectively), whilst the resonances in dimension $n = 4$ are s -resonances. It was observed in [21, 31] that no such phenomena can occur in dimension $n \geq 5$.

In [23] a precise low-energy expansion of the resolvents $R_0(z)$ and $R(z) = (H - z)^{-1}$ was given in dimension $n = 4$, providing the definition of a resonance as a solution ψ to the equation $H\psi = 0$ which is not square-integrable but lies in some weighted Sobolev space. Obtaining such an expansion in dimension $n = 2$ was a more difficult task. In [7, 8] a low energy expansion of the resolvent $R(z)$ was provided in dimension $n = 2$ to give a precise definition of resonances and determine their contribution to Levinson’s theorem. It was shown that s -resonances give no contribution to Levinson’s theorem and p -resonances give an integer contribution, both contrasting the behaviour of resonances in dimension $n = 1, 3$.

More recently in dimension $n = 4$ it was shown in [11, Equation (26b)] that in the case of a sufficiently decaying spherically symmetric potential, the s -resonances provide an integer contribution to Levinson’s theorem. In [25, Theorem 1.1] this result was generalised to non-spherically symmetric potentials.

In [24] a new symmetrised technique was introduced for performing low-energy expansions of an operator related to $R(z)$ in terms of powers of a single variable, avoiding the double Laurent expansion of [8]. The technique in [24] allows us to systematically isolate the behaviour at zero in the spectrum of H into the range of a decreasing sequence of finite rank projections. This technique is described in detail for four-dimensional Schrödinger operators in [12]. See also [14, 15, 43].

In [38, Theorem 1.1], [40, Theorem 1.3] and [4, Theorem 3.1] it was shown that for all $n \geq 2$ and suitably decaying potentials V (with the additional assumption that there are no p -resonances in dimension $n = 2$ and no resonances in dimension $n = 4$), the wave operator W_- is of the form

$$W_- = \text{Id} + \varphi(D_n)(S - \text{Id}) + K, \quad (1.2)$$

where D_n is the generator of the dilation group on $L^2(\mathbb{R}^n)$, $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ is given by

$$\varphi(x) = \frac{1}{2} \left(1 + \tanh(\pi x) - i \cosh(\pi x)^{-1} \right) \quad (1.3)$$

and K is a compact operator.

In recent years much work has been done on developing formulae analogous to Eq. (1.2) for various scattering systems, including Schrödinger scattering [4, 29, 30, 38–40], point interactions [28], rank-one perturbations [37], the Friedrichs–Faddeev

model [20], Aharonov–Bohm operators [27], lattice scattering [6], half-line scattering [19], discrete scattering [17], scattering for an inverse-square potential [10, 18] and for 1D Dirac operators with zero-range interactions [36].

The exclusion of p -resonances in dimension $n = 2$ and resonances in dimension $n = 4$ to obtain Eq. (1.2) is due to the presence of an additional logarithmic singularity near zero in the resolvent expansion provided by [12, Proposition 5.3]. Recent work [3, Lemma 4.3] shows, using the resolvent expansions of [24], that in the presence of p -resonances in dimension $n = 2$ the wave operator satisfies

$$W_- = \text{Id} + \varphi(D_n)(S - \text{Id}) + B_{\text{res}} + K, \quad (1.4)$$

where φ is given in Eq. (1.3), B_{res} is a bounded operator depending on the existence of p -resonances and K is a compact operator.

The nature of the singularity in the resolvent at zero energy in the presence of resonances in dimension $n = 4$ is the same as that of p -resonances in dimension $n = 2$. As such, we can use the techniques of [3] to show in Theorem 3.1 that Eq. (1.4) holds in dimension $n = 4$ with B_{res} an operator depending on the existence of resonances.

After factorising the wave operator as the product of two Fredholm operators, we show in Corollary 4.5 that

$$\text{Index}(W_-) = \text{Index}(W_S) + \dim(P_s), \quad (1.5)$$

where $\dim(P_s) \in \{0, 1\}$ is the number of linearly independent s -resonances and the operator $W_S = \text{Id} + \varphi(D_4)(S - \text{Id})$ is a Fredholm operator.

For a comparison, we recall that in [25, Theorem 1.1] it is proved that for a sufficiently rapidly decaying potential the number of eigenvalues N of H (counted with multiplicity) is given by

$$-N = \frac{1}{2\pi i} \int_0^\infty (\text{Tr}(S(\lambda)^* S'(\lambda)) - c_1) \, d\lambda - \beta_2 + \dim(P_s) \quad (1.6)$$

for some constants $c_1, \beta_2 \in \mathbb{C}$ depending on the potential. Even after taking into account the equality $\text{Index}(W_-) = -N$, Eq. (1.5) cannot be immediately compared with Eq. (1.6) other than to note that the existence of a resonance provides an integer contribution to both. To show that Eq. (1.6) follows from Eq. (1.5) requires a more subtle analysis of the high-energy behaviour of the scattering matrix in the trace norm, for which we defer to a future publication.

The layout of the paper is as follows. In Sect. 2 we introduce the relevant concepts from scattering theory for four-dimensional Schrödinger operators and fix our notation. In particular, we discuss the low-energy resolvent expansion provided by [12, 24], the definition of a resonance and the properties of the scattering matrix. In Sect. 3 we prove, via a number of technical results and unitary transformations analogous to those in [3], that the wave operator W_- in dimension $n = 4$ satisfies Eq. (1.4). In Sect. 4 we decouple the resonant contribution B_{res} by providing a suitable factorisation of the

wave operator and show that Eq. (1.5) holds, giving a topological interpretation of Levinson's theorem in which resonances provide an integer contribution.

2 Preliminaries on scattering theory

2.1 Standing assumptions and notation

Throughout this article we will consider the scattering theory on \mathbb{R}^4 associated to the operators

$$H_0 = - \sum_{j=1}^4 \frac{\partial^2}{\partial x_j^2} = -\Delta \quad \text{and} \quad H = - \sum_{j=1}^4 \frac{\partial^2}{\partial x_j^2} + V$$

where the (multiplication operator by the) potential V is real-valued and satisfies

$$|V(x)| \leq C(1 + |x|)^{-\rho} \quad (2.1)$$

for some $\rho > 12$. We denote the Schwartz space $\mathcal{S}(\mathbb{R}^4)$ and its dual $\mathcal{S}'(\mathbb{R}^4)$ and recall the weighted Sobolev spaces

$$H^{s,t}(\mathbb{R}^4) = \left\{ f \in \mathcal{S}'(\mathbb{R}^4) : \|f\|_{H^{s,t}} := \left\| (1 + |x|^2)^{\frac{t}{2}} (\text{Id} - \Delta)^{\frac{s}{2}} f \right\| < \infty \right\}$$

with index $s \in \mathbb{R}$ indicating derivatives and $t \in \mathbb{R}$ associated to decay at infinity [5, Section 4.1]. With $\langle \cdot, \cdot \rangle$ the Euclidean inner product on \mathbb{R}^4 , we denote the Fourier transform by

$$\mathcal{F} : L^2(\mathbb{R}^4) \rightarrow L^2(\mathbb{R}^4), \quad [\mathcal{F}f](\xi) = (2\pi)^{-2} \int_{\mathbb{R}^4} e^{-i\langle x, \xi \rangle} f(x) \, dx.$$

Note that the Fourier transform \mathcal{F} is an isomorphism from $H^{s,t}$ to $H^{t,s}$ for any $s, t \in \mathbb{R}$. We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ the bounded and compact operators from \mathcal{H}_1 to \mathcal{H}_2 . For $z \in \mathbb{C} \setminus \mathbb{R}$, we let

$$R_0(z) = (H_0 - z)^{-1}, \quad R(z) = (H - z)^{-1}$$

and the boundary values of the resolvent are defined as

$$R_0(\lambda \pm i0) = \lim_{\varepsilon \rightarrow 0} R_0(\lambda \pm i\varepsilon) \quad \text{and} \quad R(\lambda \pm i0) = \lim_{\varepsilon \rightarrow 0} R(\lambda \pm i\varepsilon). \quad (2.2)$$

The limiting absorption principle [1, Theorem 4.2] tells us that these boundary values exist in $\mathcal{B}(H^{0,t}, H^{2,-t})$ for any $t > \frac{1}{2}$ and $\lambda \in (0, \infty)$. The operator H_0 has purely absolutely continuous spectrum, and in particular no kernel. The operator H can have eigenvalues and for V satisfying Assumption (2.1) with $\rho > 1$ we have that these eigenvalues are negative, or zero [42, Theorem 6.1.1] (see also [26, Section 1]). We let P_0 be the kernel projection of H , which may be zero.

The one-parameter unitary group of dilations on $L^2(\mathbb{R}^n)$ is given on $f \in L^2(\mathbb{R}^n)$ by

$$[U_n(t)f](x) = e^{\frac{nt}{2}} f(e^t x), \quad t \in \mathbb{R}. \quad (2.3)$$

We denote the self-adjoint generator of U_n by D_n . The generator of the group $(U_+(t))$ of dilations on the half-line \mathbb{R}^+ is denoted D_+ (which is D_1 restricted to the positive half-line). The generators of the dilation groups are given by

$$D_+ = \frac{y}{i} \frac{d}{dy} + \frac{1}{2i} \text{Id}, \quad D_n = \sum_{j=1}^n \frac{x_j}{i} \frac{\partial}{\partial x_j} + \frac{n}{2i} \text{Id}. \quad (2.4)$$

Since each of D_+ , D_n generates one-parameter groups, we can recognise functions of these operators. For D_+ and $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ a bounded function whose Fourier transform has rapid decay, we have

$$[\varphi(D_+)g](\rho) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} [\mathcal{F}^* \varphi](t) e^{\frac{t}{2}} g(e^t \rho) dt,$$

with a similar formula for D_n .

Several Hilbert spaces recur, and we adopt the notation (following [22, Section 2]) which contains an excellent discussion on the relations between the spaces and operators we introduce here)

$$\mathcal{H} = L^2(\mathbb{R}^4), \quad \mathcal{P} = L^2(\mathbb{S}^3), \quad \mathcal{H}_{spec} = L^2(\mathbb{R}^+, \mathcal{P}) \cong L^2(\mathbb{R}^+) \otimes \mathcal{P}.$$

Here \mathcal{H}_{spec} provides the Hilbert space on which we can diagonalise the free Hamiltonian H_0 .

Since V is bounded, $H = H_0 + V$ is self-adjoint with $\text{Dom}(H) = \text{Dom}(H_0)$. The wave operators

$$W_{\pm} = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}$$

exist and are asymptotically complete if $\rho > 1$ [42, Theorem 1.6.2]. We will use the stationary scattering theory, which coincides with the time-dependent approach [41, Section 5.3] given our assumptions. For suitable $f, g \in \mathcal{H}$ we can write [42, Equation 0.6.9]

$$\langle W_{\pm} f, g \rangle = \int_{\mathbb{R}} \left(\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\pi} \langle R_0(\lambda \pm i\varepsilon) f, R(\lambda \pm i\varepsilon) g \rangle \right) d\lambda. \quad (2.5)$$

For our analysis of the wave operator, we describe the explicit unitaries which diagonalise our Hamiltonians.

For $\lambda > 0$ the trace operator $\gamma(\lambda) : \mathcal{S}(\mathbb{R}^4) \rightarrow \mathcal{P}$ defined by $[\gamma(\lambda)f](\omega) := f(\lambda^{\frac{1}{2}}\omega)$ defines a bounded operator and for each $s > \frac{1}{2}$ and $t \in \mathbb{R}$ extends to a bounded operator on $H^{s,t}$ (see [32, Theorem 2.4.3]).

Definition 2.1 For $\lambda \in \mathbb{R}^+$, $s \in \mathbb{R}$ and $t > \frac{1}{2}$ we define the operator

$$\Gamma_0(\lambda) : H^{s,t} \rightarrow \mathcal{P} \quad \text{by} \quad [\Gamma_0(\lambda)f](\omega) = 2^{-\frac{1}{2}}\lambda^{\frac{1}{2}}[\mathcal{F}f](\lambda^{\frac{1}{2}}\omega)$$

and the operator which diagonalises the free Hamiltonian H_0 as

$$F_0 : \mathcal{H} \rightarrow \mathcal{H}_{spec} \quad \text{by} \quad [F_0f](\lambda, \omega) = [\Gamma_0(\lambda)f](\omega).$$

Lemma 2.2 ([22, p. 439]) *The operator F_0 is unitary. Moreover, for $\lambda \in [0, \infty)$, $\omega \in \mathbb{S}^3$ and $f \in \mathcal{H}_{spec}$ we have*

$$[F_0 H_0 F_0^* f](\lambda, \omega) = \lambda f(\lambda, \omega) =: [Lf](\lambda, \omega).$$

Here we have defined the operator L of multiplication by the spectral variable.

2.2 Resolvent expansions, resonances and the scattering operator

Here we recall some known results regarding expansions related to the perturbed resolvent $R(z)$ in the limit $z \rightarrow 0$. Only the terms in the expansion relevant to later computations will be shown; however, we note that higher terms can be computed explicitly [12, 23, 43]. The low-energy behaviour is sensitive to the presence of ‘zero-energy resonances’. These are essentially distributional solutions to $H\psi = 0$ which are not square-integrable but lie in some larger space. We will give the precise definition shortly. We define $v(x) = |V(x)|^{\frac{1}{2}}$ and $u(x) = \text{sign}(V(x))$ for $x \in \mathbb{R}$, so that U, v are self-adjoint, U is unitary and $V = vUv$.

Definition 2.3 Suppose that V satisfies the assumption (2.1) for some $\rho > 12$. Then we say that there exists an s -resonance (or just simply a resonance) for $H = H_0 + V$ if there exists $\psi \in H^{0,-t}$ for some $t > 0$ such that $H\psi = 0$ (in the sense of distributions) and $\psi \notin \mathcal{H}$.

Since H has no positive eigenvalues, our assumption on the decay of the potential and the limiting absorption principle [1, Theorem 4.2] guarantee that the norm limits

$$vR_0(\lambda \pm i0)v := \lim_{\varepsilon \rightarrow 0} vR_0(\lambda \pm i\varepsilon)v \quad \text{and} \quad vR(\lambda \pm i0)v := \lim_{\varepsilon \rightarrow 0} vR(\lambda \pm i\varepsilon)v$$

exist in $\mathcal{B}(\mathcal{H})$ and are continuous in $\lambda \in (0, \infty)$. For $\lambda, \varepsilon > 0$ we have the equality

$$U - UvR(\lambda \pm i\varepsilon)vU = (U + vR_0(\lambda \pm i\varepsilon)v)^{-1},$$

which implies the existence and continuity of $(0, \infty) \ni \lambda \mapsto (u + vR_0(\lambda \pm i0)v)^{-1} \in \mathcal{B}(\mathcal{H})$. Furthermore, we have $\lim_{\lambda \rightarrow \infty} (U + vR_0(\lambda \pm i0)v)^{-1} = U$, since $\lim_{\lambda \rightarrow \infty} vR_0(\lambda \pm i0)v = 0$ by [42, Proposition 7.1.2]. On the other hand, the existence of the limits $\lim_{\lambda \rightarrow 0} (U + vR_0(\lambda \pm i0)v)^{-1}$ depends on the existence of resonances

and eigenvalues at zero. The problem has been described in detail in [12] using the method of [24]. We outline the main results below.

Let $k \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re}(k) \geq 0$ and define $\eta = \frac{1}{\ln(k)}$ (using the principal branch of the logarithm). Define the operator $M(k) = U + vR_0(-k^2)v$. Then we have the following result [12, Proposition 5.3].

Theorem 2.4 *If V satisfies assumption (2.1) with $\rho > 12$ and $0 < |k| < k_0$ for sufficiently small k_0 , the operator $M(k)^{-1}$ has the expansion*

$$M(k)^{-1} = k^{-2}D_2 + k^{-2}h(k)Q_1\tilde{T}_1Q_1 + h(k)A_1 + \eta^{-1}h(k)A_2 + \eta^{-1}Q_1A_3Q_1 + \tilde{R}(k),$$

where $h(k) = (c_1 + \eta^{-1}c_2)^{-1}$ for some $c_1, c_2 \in \mathbb{C}$, $\tilde{R}(k)$ is uniformly bounded for $0 < |k| < k_0$, $D_2, \tilde{T}_1, A_1, A_2, A_3 \in \mathcal{B}(\mathcal{H})$, $Q_1 \geq Q_2$ are orthogonal projections, $T_1 = Q_1 - Q_2$ is (at most) a rank one projection and

$$\tilde{T}_1 = T_1 - T_1C_{12}Q_2 - Q_2C_{21}T_1 + Q_2C_{22}Q_2$$

for some $C_{12}, C_{21}, C_{22} \in \mathcal{B}(\mathcal{H})$.

The projection T_1 is related to the existence of resonances. For a nonzero resonance ψ , the projection T_1 is given by $T_1 = \langle \cdot, Uv\psi \rangle Uv\psi$. The value of the constant c_1 is not important for our analysis, whilst we have

$$c_2 = -\frac{(\langle v, Uv\psi \rangle)^2}{(8\pi^2)}, \quad (2.6)$$

see, for example, [2, Lemma 3.2.36] and [43, Equation 2.2]. To complete the analysis, we also need the small energy behaviour of $\Gamma_0(\lambda)v$, which we can obtain from [4, Lemma 2.12].

Lemma 2.5 *Suppose that V satisfies (2.1) for some $\rho > 12$. Then in $\mathcal{B}(\mathcal{H}, \mathcal{P})$ we have the expansion*

$$\Gamma_0(\lambda)v = \lambda^{\frac{1}{2}}\gamma_0v + \lambda\gamma_1v + O(\lambda^{\frac{3}{2}})$$

as $\lambda \rightarrow 0$ in $\mathcal{B}(\mathcal{H}, \mathcal{P})$, where the operators $\gamma_0v, \gamma_1v \in \mathcal{B}(\mathcal{H}, \mathcal{P})$ are given for $f \in \mathcal{H}$ by

$$[\gamma_jvf](\omega) = 2^{-\frac{1}{2}}(2\pi)^{-2} \int_{\mathbb{R}^4} (-i\langle x, \omega \rangle)^j v(x) f(x) dx.$$

We can combine Theorem 2.4 and Lemma 2.5 to obtain an expansion of a related operator which will be useful in the next section. We use the convention that $\lambda > 0$, $k = -i\lambda^{\frac{1}{2}}$ and $\eta = \left(\frac{\ln(\lambda)}{2} - i\frac{\pi}{2}\right)^{-1}$.

Lemma 2.6 *Suppose V satisfies the assumption (2.1) for some $\rho > 12$. Then as $\lambda \rightarrow 0$ we have the expansion*

$$(U + vR_0(\lambda + i0)v)^{-1} v\Gamma_0(\lambda)^* = \lambda^{-\frac{1}{2}} h(\lambda^{\frac{1}{2}})(T_1 - Q_2 C_{21} T_1) v\gamma_0^* + D_2 v\gamma_1^* + O(\eta)$$

in $\mathcal{B}(\mathcal{P}, \mathcal{H})$.

Proof The identity $\gamma_0 v Q_2 = 0 = Q_2 v \gamma_0^*$ [12, Lemma 7.2] yields the equality $\tilde{T}_1 v \gamma_0^* = (T_1 - Q_2 C_{21} T_1) v \gamma_0^*$. Observing that as $\lambda \rightarrow 0$, $\eta \rightarrow 0$ and $h(\lambda^{\frac{1}{2}}) \rightarrow 0$, we multiply the expansions of Theorem 2.4 and Lemma 2.5 to obtain the statement. \square

We summarise below some useful properties of the scattering matrix [42, Proposition 1.8.1 and Proposition 8.1.9], [4, Theorem 2.15 and Corollary 3.10].

Theorem 2.7 *Suppose that V satisfies the assumption (2.1) for some $\rho > 1$. The scattering matrix $S(\lambda)$ is given for all $\lambda \in \mathbb{R}^+$ by the equation*

$$S(\lambda) = \text{Id} - 2\pi i \Gamma_0(\lambda) v (U + vR_0(\lambda + i0)v)^{-1} v\Gamma_0(\lambda)^*. \quad (2.7)$$

For each $\lambda \in \mathbb{R}^+$, the operator $S(\lambda)$ is unitary in $\mathcal{P} = L^2(\mathbb{S}^3)$ and depends continuously (in the sense of norm) on $\lambda \in \mathbb{R}^+$. Furthermore, if $\rho > 12$ then $S(\lambda) - \text{Id} \in \mathcal{L}^1(\mathcal{H})$, is differentiable in the norm of $\mathcal{L}^1(\mathcal{H})$ and we have $S(0) = \text{Id}$ and $\lim_{\lambda \rightarrow \infty} S(\lambda) = \text{Id}$, where the limit is taken in $\mathcal{B}(\mathcal{P})$.

3 The form of the wave operator

In this section, we analyse the wave operator W_- in the space $\mathcal{H}_{\text{spec}}$ (the spectral representation for H_0). Much of the analysis is identical to the two-dimensional case in [3], to which we refer for many details. The main result in this section is the following.

Theorem 3.1 *Suppose that V satisfies the estimate (2.1) for some $\rho > 12$. Then the wave operator W_- is given by*

$$W_- = \text{Id} + \varphi(D_4)(S - \text{Id}) + B_{\text{res}} + K, \quad (3.1)$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ is given by

$$\varphi(x) = \frac{1}{2} \left(1 + \tanh(\pi x) - i \cosh(\pi x)^{-1} \right),$$

B_{res} is a bounded operator depending on the existence of resonances and K is a compact operator.

In Lemma 3.5 we will determine an explicit expression for the operator B_{res} in a different representation. We begin our analysis of the wave operator with the stationary representation for the wave operator W_- . For suitable $f, g \in \mathcal{H}_{spec}$ we have

$$\begin{aligned} & \langle F_0(W_- - \text{Id})F_0^*f, g \rangle \\ &= - \int_{\mathbb{R}} \lim_{\varepsilon \rightarrow 0} \left(\int_0^\infty \left\langle \Gamma_0(\mu)v (U + vR_0(\lambda + i\varepsilon)v)^{-1} vF_0^*\delta_\varepsilon(L - \lambda)f, \frac{g(\mu)}{\mu - \lambda + i\varepsilon} \right\rangle_p d\mu \right) d\lambda, \end{aligned} \quad (3.2)$$

where we have defined

$$\delta_\varepsilon(L - \lambda) := \frac{\varepsilon}{\pi} (L - \lambda + i\varepsilon)^{-1} (L - \lambda - i\varepsilon)^{-1}.$$

Next we write $\Gamma_0(\mu)v = \Gamma_0(\mu)vQ_1 + \Gamma_0(\mu)vQ_1^\perp$, which then gives two terms in Eq. (3.2). The term with Q_1^\perp has no singularity at zero energy by Lemma 2.6 and has been considered in [4]. A similar expansion has been considered in great detail in [3] in dimension $n = 2$.

We now carefully analyse Eq. (3.2). Fix a function $\chi_1 \in C(\mathbb{R}^+; [0, 1])$ with $\chi_1(\lambda) = 0$ for $\lambda < \frac{1}{4}$ and $\chi_1(\lambda) = 1$ for $\lambda > \frac{3}{4}$ and let $\chi_2 = 1 - \chi_1$. Let also the function $\chi_0 \in C(\mathbb{R}^+; [0, 1])$ be such that $\chi_0(\lambda) = 1$ for $\lambda < \frac{3}{4}$ and $\chi_0(\lambda) = 0$ for $\lambda > \frac{7}{8}$. For $f \in \mathcal{H}_{spec}$ and $\lambda, \varepsilon > 0$ we define

$$\tilde{\psi}_\varepsilon(\lambda) := Q_1 (U + vR_0(\lambda + i\varepsilon)v)^{-1} vF_0^*\delta_\varepsilon(L - \lambda)\chi_2(L)f.$$

We make the following decomposition of Eq. (3.2). Define

$$\begin{aligned} R_0 &= \int_0^\infty \lim_{\varepsilon \rightarrow 0} \left(\int_0^\infty \left\langle \frac{\Gamma_0(\mu)vQ_1}{\mu - \lambda - i\varepsilon} Q_1 (U + vR_0(\lambda + i\varepsilon)v)^{-1} vF_0^*\delta_\varepsilon(L - \lambda)\chi_1(L)f, g(\mu) \right\rangle d\mu \right) d\lambda, \\ R_1 &= \int_0^\infty \lim_{\varepsilon \rightarrow 0} \left(\int_0^\infty \left\langle \Gamma_0(\mu)vQ_1\chi_2(\mu)\mu^{-\frac{1}{2}}(\mu^{\frac{1}{2}} - \lambda^{\frac{1}{2}})(\mu - \lambda - i\varepsilon)^{-1} vF_0^*\tilde{\psi}_\varepsilon(\lambda), g(\mu) \right\rangle d\mu \right) d\lambda, \\ R_2 &= \int_0^\infty \lim_{\varepsilon \rightarrow 0} \left(\int_0^\infty \left\langle \Gamma_0(\mu)vQ_1\chi_2(\mu)\mu^{-\frac{1}{2}}\lambda^{\frac{1}{2}}(\mu - \lambda - i\varepsilon)^{-1} vF_0^*\tilde{\psi}_\varepsilon(\lambda), g(\mu) \right\rangle d\mu \right) d\lambda, \\ R_3 &= \int_0^\infty \lim_{\varepsilon \rightarrow 0} \left(\int_0^\infty \left\langle \Gamma_0(\mu)vQ_1\chi_1(\mu)\mu^{-1}(\mu - \lambda)(\mu - \lambda - i\varepsilon)^{-1} vF_0^*\tilde{\psi}_\varepsilon(\lambda), g(\mu) \right\rangle d\mu \right) d\lambda, \\ R_4 &= \int_0^\infty \lim_{\varepsilon \rightarrow 0} \left(\int_0^\infty \left\langle \Gamma_0(\mu)vQ_1\chi_1(\mu)\mu^{-1}\lambda(\mu - \lambda - i\varepsilon)^{-1} vF_0^*\tilde{\psi}_\varepsilon(\lambda), g(\mu) \right\rangle d\mu \right) d\lambda, \text{ and} \\ R_5 &= \int_0^\infty \lim_{\varepsilon \rightarrow 0} \left(\int_0^\infty \left\langle \Gamma_0(\mu)vQ_1^\perp\mu^{-1}(\mu - \lambda)(\mu - \lambda - i\varepsilon)^{-1} vF_0^*\tilde{\psi}_\varepsilon(\lambda), g(\mu) \right\rangle d\mu \right) d\lambda. \end{aligned}$$

By construction we have the equality

$$-\langle F_0(W_- - \text{Id})F_0^*f, g \rangle = R_0 + R_1 + R_2 + R_3 + R_4 + R_5.$$

Combining [3, Corollary 4.2] and [4, Theorem 3.1] we have the following.

Lemma 3.2 Suppose that V satisfies (2.1) for some $\rho > 12$. Then we have

$$R_0 + R_2 + R_3 + R_4 + R_5 = -\langle (\text{Id} + \varphi(D_4)(S - \text{Id}) + K) f, g \rangle$$

for a compact operator K .

It remains to analyse the term R_1 , which contains the worst singularity of $M(k)^{-1}$. For this analysis we require some additional preparatory results. We consider the unitary transformation $\mathcal{U} : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R})$ defined for $f \in L^2(\mathbb{R}^+)$ and $x \in \mathbb{R}$ by

$$[\mathcal{U}f](x) = 2^{-\frac{1}{2}} e^{-x} f(e^{-2x}). \quad (3.3)$$

We also introduce the integral operator $\Xi : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ with kernel given by

$$\Xi(\mu, \lambda) = \chi_0(\mu) \frac{1}{\mu^{\frac{1}{2}} + \lambda^{\frac{1}{2}}} \frac{1}{\lambda^{\frac{1}{2}} \ln(\lambda)} \chi_0(\lambda). \quad (3.4)$$

We can identify Ξ as a function of the operators X of position and D_1 of dilation in \mathbb{R} . The following is [3, Lemma 3.6].

Lemma 3.3 Define $\Xi : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ by Eq. (3.4). Then we have in $L^2(\mathbb{R})$ the equality

$$\mathcal{U}\Xi\mathcal{U}^* = -\chi_0(e^{-2x}) \frac{2}{1 + 2iD_1} \chi_0(e^{-2x}) + K_1,$$

where D_1 is the generator of dilations in $L^2(\mathbb{R})$, X is the operator of multiplication by the variable in $L^2(\mathbb{R})$ and K_1 is a compact operator.

To analyse the term R_1 we need to introduce two more operators, analogues of which have been studied in [3, Lemmas 3.1 and 3.5].

Lemma 3.4 Define the maps $B : \mathbb{R}^+ \rightarrow \mathcal{K}(\mathcal{P}, \mathcal{H})$ and $N : \mathbb{R}^+ \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{P})$ for $\lambda, \mu \in \mathbb{R}^+$ by

$$B(\lambda) = \chi_2(\lambda) \lambda^{\frac{1}{2}} \ln(\lambda) Q_1 (U + vR_0(\lambda + i0)v)^{-1} v\Gamma_0(\lambda)^*, \quad \text{and} \\ N(\mu) = \Gamma_0(\mu) \chi_2(\mu) \mu^{-\frac{1}{2}} v Q_1.$$

The multiplication operator defined by B extends continuously to $\mathcal{B}(\mathcal{H}_{\text{spec}}, L^2(\mathbb{R}^+, \mathcal{H}))$ and the multiplication operator defined by N extends continuously to a bounded operator in $\mathcal{B}(L^2(\mathbb{R}^+, \mathcal{H}), \mathcal{H}_{\text{spec}})$.

By an identical proof to [3, Lemma 3.7] we obtain the following.

Lemma 3.5 The term R_1 can be written as $\langle (B_{\text{res}} + K)f, g \rangle$ with $B_{\text{res}} = N\Xi B$ and K compact.

Collecting together the results of Lemmas 3.2 and 3.5 we can now prove Theorem 3.1.

Proof of Theorem 3.1 By construction we have the equality

$$-\langle F_0(W_- - \text{Id})F_0^*f, g \rangle = R_0 + R_1 + R_2 + R_3 + R_4 + R_5.$$

By Lemma 3.2 we have

$$R_0 + R_2 + R_3 + R_4 + R_5 = -\langle (\text{Id} + \varphi(D_4)(S - \text{Id}) + K)f, g \rangle.$$

By Lemma 3.5 we have $R_1 = \langle (B_{res} + K)f, g \rangle$ with $B_{res} = N\Xi B$ and K compact. Adding these results together gives

$$F_0(W_- - \text{Id})F_0^* = \varphi(D_4)(S - \text{Id}) - N\Xi B + K \quad (3.5)$$

for a compact operator K , from which the statement follows. \square

As suggested by the analysis of Lemma 3.3, we investigate Eq. (3.5) in the Hilbert space $L^2(\mathbb{R}, \mathcal{P})$ by using the unitary map \mathcal{U} of Eq. (3.3). For any multiplication operator M defined by $\mathbb{R}^+ \ni \lambda \mapsto M(\lambda)$ and $f \in L^2(\mathbb{R}, \mathcal{P})$ we have

$$[\mathcal{U}M\mathcal{U}^*f](x) = M(e^{-2x})f(x) = [M(e^{-2X})f](x).$$

If we consider the dilation group $(U_+(t))_{t \in \mathbb{R}}$ we obtain

$$[\mathcal{U}U_+(t)\mathcal{U}^*f](x) = f\left(x - \frac{1}{2}t\right) = [e^{-it\frac{1}{2}D}f](x),$$

where $D = -i\frac{d}{dx}$. Combining this information we find

$$\begin{aligned} & \mathcal{U}F_0(W_- - \text{Id})F_0^*\mathcal{U}^* \\ &= \left(\frac{1}{2}(\text{Id} - \tanh\left(\frac{\pi}{2}D\right) - i \cosh\left(\frac{\pi}{2}D\right)^{-1}\right)(\tilde{S}(X) - \text{Id}) \\ & \quad - \tilde{N}(X)\left(\frac{2}{1 + 2iD_1}\right)\tilde{B}(X) + K, \end{aligned}$$

where $\tilde{S}(X) = S(e^{-2X})$, $\tilde{N}(X) = N(e^{-2X})$, $\tilde{B}(X) = B(e^{-2X})$ and K is a compact operator. We note that the three generators X , D , D_1 , the position operator, the generator of translations and the generator of dilations are all present in this expression. A C^* -algebra generated by continuous functions of these three operators has been considered in [9, Chapter V]. The algebra of Cordes is constructed on the Hilbert space $L^2(\mathbb{R}^+)$, so we need an additional unitary to use this framework, namely the decomposition of $L^2(\mathbb{R})$ into even and odd functions.

Define $\mathcal{V} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^+, \mathbb{C}^2)$ by

$$[\mathcal{V}f](y) = 2^{\frac{1}{2}} \begin{pmatrix} \frac{f(y) + f(-y)}{2} \\ \frac{f(y) - f(-y)}{2} \end{pmatrix} := 2^{\frac{1}{2}} \begin{pmatrix} f_e \\ f_o \end{pmatrix}$$

for $f \in L^2(\mathbb{R})$. The adjoint of \mathcal{V} is defined for $g_1, g_2 \in L^2(\mathbb{R}^+)$ by

$$\left[\mathcal{V}^* \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right] (x) = 2^{-\frac{1}{2}} \left[g_1(|x|) + \text{sign}(x) g_2(|x|) \right].$$

If $m : \mathbb{R} \rightarrow \mathbb{C}$ we find

$$\mathcal{V}m(X)\mathcal{V}^* = \begin{pmatrix} m_e(L) & m_o(L) \\ m_o(L) & m_e(L) \end{pmatrix},$$

whilst

$$\mathcal{V}m(D_1)\mathcal{V}^* = \begin{pmatrix} m(D_+) & 0 \\ 0 & m(D_+) \end{pmatrix}.$$

In order to consider $\mathcal{V}m(D)\mathcal{V}^*$, let us denote by \mathcal{F}_1 the usual unitary Fourier transform in $L^2(\mathbb{R})$, and let $\mathcal{F}_N, \mathcal{F}_D$ be the unitary cosine and sine transforms on $L^2(\mathbb{R}_+)$, respectively. The index N and D are related to the Neumann Laplacian and the Dirichlet Laplacian in $L^2(\mathbb{R}_+)$, which are diagonalised by \mathcal{F}_N and \mathcal{F}_D , respectively. Note also that the operators \mathcal{F}_N and \mathcal{F}_D correspond to their own inverse. It is then easily checked that

$$\mathcal{V}\mathcal{F}_1\mathcal{V}^* = \begin{pmatrix} \mathcal{F}_N & 0 \\ 0 & i\mathcal{F}_D \end{pmatrix}.$$

In addition, by a straightforward computation one gets

$$\mathcal{V}m(D)\mathcal{V}^* = \mathcal{V}\mathcal{F}_1^*m(X)\mathcal{F}_1\mathcal{V}^* = \begin{pmatrix} \mathcal{F}_Nm_e(L)\mathcal{F}_N & -i\mathcal{F}_Nm_o(L)\mathcal{F}_D \\ i\mathcal{F}_Dm_o(L)\mathcal{F}_N & \mathcal{F}_Dm_e(L)\mathcal{F}_D \end{pmatrix}.$$

For the final step, we recall that the Neumann Laplacian satisfies $-\Delta_N := \mathcal{F}_N L^2 \mathcal{F}_N$, and that

$$i\mathcal{F}_N\mathcal{F}_D = -\tanh(\pi D_+) + i \cosh(\pi D_+)^{-1} =: \phi(D_+).$$

We refer for example to [10, Proposition 4.13] for a proof of the above equality. Then, we obtain

$$\mathcal{V}m(D)\mathcal{V}^* = \mathcal{V}\mathcal{F}_1^*m(X)\mathcal{F}_1\mathcal{V}^* = \begin{pmatrix} m_e(\sqrt{-\Delta_N}) & -m_o(\sqrt{-\Delta_N})\phi(D_+) \\ -\bar{\phi}(D_+)m_o(\sqrt{-\Delta_N}) & \bar{\phi}(D_+)m_e(\sqrt{-\Delta_N})\phi(D_+) \end{pmatrix}.$$

Combining these we can obtain the image of the wave operator in $L^2(\mathbb{R}^+, \mathbb{C}^2)$.

Lemma 3.6 *The expression $\mathcal{V}\mathcal{U}F_0W_-F_0^*\mathcal{U}^*\mathcal{V}^*$ is given by*

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & \bar{\phi} \left(\frac{1}{2} \sqrt{-\Delta_N} \right) \phi(D_+) \\ \phi(D_+) \bar{\phi} \left(\frac{1}{2} \sqrt{-\Delta_N} \right) & 1 \end{pmatrix} \begin{pmatrix} \tilde{S}_e(L) - 1 & \tilde{S}_o(L) \\ \tilde{S}_o(L) & \tilde{S}_e(L) - 1 \end{pmatrix} \\ & + \begin{pmatrix} \tilde{N}_e(L) & \tilde{N}_o(L) \\ \tilde{N}_o(L) & \tilde{N}_e(L) \end{pmatrix} \begin{pmatrix} \frac{2}{1+2iD_+} & 0 \\ 0 & \frac{2}{1+2iD_+} \end{pmatrix} \begin{pmatrix} \tilde{B}_e(L) & \tilde{B}_o(L) \\ \tilde{B}_o(L) & \tilde{B}_e(L) \end{pmatrix} + K \end{aligned}$$

with K compact.

We now determine the precise contribution of resonances to the above equality.

Lemma 3.7 *We have the equality*

$$N(0)B(0) = -P_s,$$

where $P_s = 0$ if there does not exist a resonance and P_s is the orthogonal projection onto the spherical harmonics of order zero in \mathcal{P} if there exists a resonance.

Proof The case when there are no resonances is clear since $T_1 = 0$, so suppose there does exist a resonance. We use Lemma 2.6 to see that

$$\begin{aligned} B(0) &= \lim_{\lambda \rightarrow 0} \chi_2(\lambda) \lambda^{\frac{1}{2}} \ln(\lambda) Q_1 (U + vR_0(\lambda + i0)v)^{-1} v\Gamma_0(\lambda)^* \\ &= \lim_{\lambda \rightarrow 0} \chi_2(\lambda) \lambda^{\frac{1}{2}} \ln(\lambda) \left(\lambda^{-\frac{1}{2}} h(\lambda^{\frac{1}{2}}) (T_1 - Q_2 C_{21} T_1) v \gamma_0^* + D_2 v \gamma_1^* + O(\eta) \right) \\ &= \left(\lim_{\lambda \rightarrow 0} \ln(\lambda) h(\lambda^{\frac{1}{2}}) \right) (T_1 - Q_2 C_{21} T_1) v \gamma_0^* \\ &= \left(\lim_{\lambda \rightarrow 0} \frac{\ln(\lambda)}{c_1 + \left(\frac{\ln(\lambda)}{2} - i \frac{\pi}{2} \right) c_2} \right) (T_1 - Q_2 C_{21} T_1) v \gamma_0^* \\ &= \frac{2}{c_2} (T_1 - Q_2 C_{21} T_1) v \gamma_0^*. \end{aligned}$$

The small energy behaviour of $\Gamma_0(\mu)v$ and the definition of N gives $N(0) = \gamma_0 v Q_1$. Observing the relation $\gamma_0 v Q_2 = 0$, we multiply to obtain

$$N(0)B(0) = \frac{2}{c_2} \gamma_0 v T_1 v \gamma_0^*.$$

Recall from the discussion around Theorem 2.4 that for a nonzero resonance ψ we have $T_1 = \langle \cdot, Uv\psi \rangle Uv\psi$. We find for $f \in \mathcal{P}$ and $\omega \in \mathbb{S}^3$ that

$$\begin{aligned} [\gamma_0 v T_1 v \gamma_0^* f](\omega) &= 2^{-\frac{1}{2}} (2\pi)^{-2} \int_{\mathbb{R}^4} v(x) [T_1 v \gamma_0^* f](x) dx \\ &= 2^{-\frac{1}{2}} (2\pi)^{-2} \int_{\mathbb{R}^4} v(x) \langle v \gamma_0^* f, Uv\psi \rangle U(x) v(x) \psi(x) dx \\ &= 2^{-1} (2\pi)^{-4} (\langle v, Uv\psi \rangle)^2 \int_{\mathbb{S}^3} f(\theta) d\theta. \end{aligned}$$

Now note that $\text{Vol}(\mathbb{S}^3) = 2\pi^2$ to see that

$$\begin{aligned} [\gamma_0 v T_1 v \gamma_0^* f](\omega) &= 2^{-1} (2\pi)^{-4} (\langle v, U v \psi \rangle)^2 (2\pi^2) \left(\frac{1}{\text{Vol}(\mathbb{S}^3)} \int_{\mathbb{S}^3} f(\theta) d\theta \right) \\ &= \frac{(\langle v, U v \psi \rangle)^2}{16\pi^2} [P_s f](\omega), \end{aligned}$$

where P_s denotes the projection onto the spherical harmonics of order zero in \mathcal{P} . Recalling the value of c_2 from Eq. (2.6), we find the statement of the lemma. \square

4 Index theory and Levinson's theorem

In [9, Section V.7], the following C^* -subalgebra of $\mathcal{B}(L^2(\mathbb{R}^+))$ was introduced:

$$E := C^*(a_i(D_+)b_i(L)c_i(-\Delta_N) : a_i \in C([-\infty, \infty]), b_i, c_i \in C([0, \infty])).$$

It is shown in [9, Theorem V.7.3] that the quotient algebra $E/\mathcal{K}(L^2(\mathbb{R}^+))$ is isomorphic to $C(\diamond)$, the set of continuous functions defined on the edges of a hexagon.

As a consequence of the results of Cordes, we have the short exact sequence

$$0 \longrightarrow \mathcal{K}(L^2(\mathbb{R}^+; \mathcal{P})^{\oplus 2}) \longrightarrow (M_2(E) \otimes \mathcal{K}(\mathcal{P}))^\sim \xrightarrow{q} (M_2(C(\diamond)) \otimes \mathcal{K}(\mathcal{P}))^\sim \longrightarrow 0$$

of C^* -algebras, and the corresponding 6 term exact sequence in K -theory. In particular, we have $K_0(\mathcal{K}(L^2(\mathbb{R}^+; \mathcal{P})^{\oplus 2})) \cong \mathbb{Z}$ and $K_1((M_2(C(\diamond)) \otimes \mathcal{K}(\mathcal{P}))^\sim) \cong \mathbb{Z}$.

This picture will allow us to interpret Levinson's theorem as a topological result. Our interest in the C^* -algebra E is that the wave operator is a product of functions of D_+ , L , $-\Delta_N$, as shown in Lemma 3.6, and these functions are continuous and have limits at their endpoints. As such, the wave operator is an element of $(M_2(E) \otimes \mathcal{K}(\mathcal{P}))^\sim$. We can then consider the image of the wave operator under the quotient map

$$q : (M_2(E) \otimes \mathcal{K}(\mathcal{P}))^\sim \rightarrow (M_2(C(\diamond)) \otimes \mathcal{K}(\mathcal{P}))^\sim,$$

with kernel $M_2(\mathcal{K}(L^2(\mathbb{R}^+))) \otimes \mathcal{K}(\mathcal{P})$.

Proposition 4.1 *The operator $\mathcal{W}U F_0 W_- F_0^* \mathcal{U}^* \mathcal{V}^*$ is an element of $(M_2(E) \otimes \mathcal{K}(\mathcal{P}))^\sim$. Hence, up to unitary equivalence, the image of the wave operator W_- through the quotient map q is a continuous function*

$$\Gamma := (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6) : \diamond \rightarrow M_2 \otimes \mathcal{K}(\mathcal{P})^\sim$$

even at the vertices of the hexagon. The restrictions to the edges (oriented as indicated) are

$$\begin{aligned}\Gamma_1(s) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(S(1) - 1) \begin{pmatrix} 1 & \phi(s) \\ \phi(s) & 1 \end{pmatrix}, & s \in [-\infty, \infty], \\ \Gamma_2(\ell) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(S(e^{2\ell}) - 1) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, & \ell \in [0, \infty], \\ \Gamma_3(\xi) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \xi \in [\infty, 0], \\ \Gamma_4(s) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2} \frac{2}{1 + 2is} P_s \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & s \in [\infty, -\infty], \\ \Gamma_5(\xi) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \xi \in [0, \infty], \\ \Gamma_6(\ell) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(S(e^{-2\ell}) - 1) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \ell \in [\infty, 0].\end{aligned}$$

Proof The continuity and existence of the limits of the endpoints of the components of $\mathcal{V}\mathcal{U}F_0W_-F_0^*\mathcal{U}^*\mathcal{V}^*$ have already been established in Lemmas 3.3 and 3.4. Note also that by Theorem 2.7 we have $\lambda \mapsto S(\lambda) - \text{Id}$ belongs to $C_0(\mathbb{R}^+, \mathcal{K}(\mathcal{P}))$. Thus, we find $\mathcal{V}\mathcal{U}F_0W_-F_0^*\mathcal{U}^*\mathcal{V}^* \in (M_2(E) \otimes \mathcal{K}(\mathcal{P}))^\sim$.

We now consider the image of the operator $\mathcal{V}\mathcal{U}F_0W_-F_0^*\mathcal{U}^*\mathcal{V}^*$ under the quotient map. For Γ_1 , we note that $\tilde{S}_e(0) = S(1)$ and $\tilde{S}_o(0) = 0$. Since χ_2 vanishes at infinity we observe also that $\tilde{N}(0) = \tilde{B}(0) = 0$. For Γ_2 , we note that $\lim_{s \rightarrow \infty} \phi(s) = -1$ to obtain the result. For Γ_6 , we use the observation that $\lim_{s \rightarrow -\infty} \phi(s) = 1$. For Γ_3 and Γ_5 , we recall that $S(0) = \lim_{\lambda \rightarrow \infty} (S(\lambda)) = \text{Id}$ to see that $\lim_{\lambda \rightarrow \pm\infty} \tilde{S}_e(\ell) = 1$, whilst $\lim_{\lambda \rightarrow \pm\infty} \tilde{S}_o(\ell) = 0$. Finally, for Γ_4 we note that $\lim_{\ell \rightarrow \infty} \tilde{N}(\ell) = N(0)$ and $\lim_{\ell \rightarrow \infty} \tilde{B}(\ell) = B(0)$. Thus, we find

$$\lim_{\ell \rightarrow \infty} \tilde{N}_e(\ell) = \lim_{\ell \rightarrow \infty} \tilde{N}_o(\ell) = \frac{1}{2}N(0)$$

and

$$\lim_{\ell \rightarrow \infty} \tilde{B}_e(\ell) = \lim_{\ell \rightarrow \infty} \tilde{B}_o(\ell) = \frac{1}{2}B(0).$$

An application of Lemma 3.7 then completes the proof. \square

As a result of [3, Lemma 5.1] we have the following.

Lemma 4.2 *The pointwise determinant of each component Γ_j exists, and they are given by*

$$\begin{aligned}\det(\Gamma_1(s)) &= \det(S(1)), & s &\in [-\infty, \infty], \\ \det(\Gamma_2(\ell)) &= \det(S(e^{2\ell})), & \ell &\in [0, \infty], \\ \det(\Gamma_3(\xi)) &= 1, & \xi &\in [\infty, 0], \\ \det(\Gamma_4(s)) &= \left(\frac{2is - 1}{2is + 1} \right)^{\dim(P_s)}, & s &\in [\infty, -\infty], \\ \det(\Gamma_5(\xi)) &= 1, & \xi &\in [0, \infty], \\ \det(\Gamma_6(\ell)) &= \det(S(e^{-2\ell})), & \ell &\in [\infty, 0].\end{aligned}$$

We now decouple the resonant contribution to the wave operator. We do this in order to isolate the numerical contribution of resonances to Levinson's theorem.

Define the operators $W_S, W_R \in \mathcal{B}(\mathcal{H})$ by the equalities

$$F_0(W_S - \text{Id})F_0^* = \frac{1}{2}\varphi\left(-\frac{1}{2}D_+\right)(S(L) - \text{Id}). \quad (4.1)$$

and

$$F_0(W_R - \text{Id})F_0^* = -N\Xi B.$$

Then we have the following.

Lemma 4.3 *The operators W_S and W_R are Fredholm, and we have the equality*

$$W_- = W_S W_R + K \quad (4.2)$$

for a compact operator K . In particular, we have

$$\text{Index}(W_-) = \text{Index}(W_S) + \text{Index}(W_R). \quad (4.3)$$

Proof To see that W_S defines a Fredholm operator, it is sufficient to note that W_{S^*} defines an inverse for W_S up to compacts (see [3, Lemma 5.3]). As in the proof of [3, Proposition 5.5] we have that W_R^* defines an inverse for W_R modulo compacts. We next observe that by Eq. (3.5) we have the equality

$$W_- = W_S + (W_R - \text{Id}) + K = W_S(\text{Id} + W_S^*(W_R - \text{Id})) + \tilde{K} \quad (4.4)$$

with K, \tilde{K} compact operators. Denote by $\Gamma_{S,j}$ and $\Gamma_{R,j}$ the components of $\mathcal{W}U F_0 W_S F_0^* \mathcal{U}^* \mathcal{V}^*$ and $\mathcal{W}U F_0 W_R F_0^* \mathcal{U}^* \mathcal{V}^*$ under the quotient map q . A proof simi-

lar to Proposition 4.1 shows that

$$\begin{aligned}\Gamma_{S,1}(s) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(S(1) - 1) \begin{pmatrix} 1 & \phi(s) \\ \phi(s) & 1 \end{pmatrix}, & s \in [-\infty, \infty], \\ \Gamma_{S,2}(\ell) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(S(e^{2\ell}) - 1) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, & \ell \in [0, \infty], \\ \Gamma_{S,3}(\xi) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \xi \in [\infty, 0], \\ \Gamma_{S,4}(s) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & s \in [\infty, -\infty], \\ \Gamma_{S,5}(\xi) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \xi \in [0, \infty], \\ \Gamma_{S,6}(\ell) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(S(e^{-2\ell}) - 1) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \ell \in [\infty, 0]\end{aligned}$$

and

$$\begin{aligned}\Gamma_{R,4}(s) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \frac{2}{1 + 2is} P_s \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & s \in [\infty, -\infty], \\ \Gamma_{R,j} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & j \in \{1, 2, 3, 5, 6\}.\end{aligned}$$

Explicit computation then shows

$$\Gamma_{S,j}^* \left(\Gamma_{R,j} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \Gamma_{R,j} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular we find

$$\Gamma_{S,j}^* \left(\Gamma_{R,j} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

if $j \in \{1, 2, 3, 5, 6\}$ and

$$\Gamma_{S,4}^*(s) \left(\Gamma_{R,4}(s) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = -\frac{1}{2} \frac{2}{1 + 2is} P_s \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

for $s \in [\infty, -\infty]$. Thus we find

$$q(\mathcal{WU}F_0(W_S^*(W_R - \text{Id}))F_0^*\mathcal{U}^*\mathcal{V}^*) = q(\mathcal{WU}F_0(W_R - \text{Id})F_0^*\mathcal{U}^*\mathcal{V}^*).$$

Since their images under the quotient map agree, we have $W_R = \text{Id} + W_S^*(W_R - \text{Id}) \pmod{\text{compacts}}$. The result then follows from Equation (4.4). \square

Lemma 4.4 *We have the equality*

$$\text{Index}(W_R) = \dim(P_s).$$

Proof The operator $q(\mathcal{V}\mathcal{U}F_0W_RF_0^*\mathcal{U}^*\mathcal{V}^*)$ has components

$$\begin{aligned}\Gamma_{R,4}(s) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2} \frac{2}{1+2is} P_s \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & s \in [\infty, -\infty], \\ \Gamma_{R,j} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & j \in \{1, 2, 3, 5, 6\}.\end{aligned}$$

The index of W_R can then be computed using Gohberg–Kreĭn theory [13] as the sum of winding numbers

$$\text{Index}(W_R) = \sum_{j=1}^6 \text{Wind}(\Gamma_{R,j}).$$

For $j \in \{1, 2, 3, 5, 6\}$ we have $\text{Wind}(\Gamma_{R,j}) = 0$. If $\dim(P_s) = 0$ we have $\text{Wind}(\Gamma_{R,4}) = 0$ also. If $\dim(P_s) = 1$, then we find

$$\begin{aligned}\text{Wind}(\Gamma_{R,4}) &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\frac{d}{ds} \det(\Gamma_{R,4}(s))}{\det(\Gamma_{R,4}(s))} ds \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{4i}{4s^2 + 1} ds \\ &= 1,\end{aligned}$$

from which the result follows. \square

Corollary 4.5 *We have the equality*

$$\text{Index}(W_-) = \text{Index}(W_S) + \dim(P_s).$$

To determine an analytic formula for $\text{Index}(W_S)$ requires a more subtle analysis of the high-energy behaviour of the scattering matrix (see for example [16, Theorem III.1] and [42, Section 9.2]) and we defer the proof to a future publication. The analytic formula has previously been obtained by [25] and reads

$$\text{Index}(W_-) = -N = \frac{1}{2\pi i} \int_0^\infty (\text{Tr}(S(\lambda)^* S'(\lambda)) - c_1) d\lambda + \dim(P_s) - \beta_2. \quad (4.5)$$

Here the constants are given by

$$c_1 = -\frac{(2\pi i) \text{Vol}(\mathbb{S}^3)}{2(2\pi)^4} \int_{\mathbb{R}^4} V(x) dx \quad \text{and} \quad \beta_2 = -\frac{\text{Vol}(\mathbb{S}^3)}{4(2\pi)^4} \int_{\mathbb{R}^4} V(x)^2 dx.$$

The proof in [25] used high-energy asymptotics of the spectral shift function, obtained using heat kernel methods. Our approach also uses high-energy asymptotics for the spectral shift, but then utilises regularised determinants, the limiting absorption principle and Gohberg–Kreĭn theory.

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Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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