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ORIGINAL PAPER

Spectral flow and Levinson's theorem for Schrödinger operators

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Abstract

We use spectral flow to present a new proof of Levinson's theorem for Schrödinger operators on $\mathbb{R}^n$ with smooth compactly supported potential. Our proof is valid in all dimensions and in the presence of resonances. The statement is expressed in terms of the spectral shift function and the "high energy corrected time delay" following Guillopé and others.

Keywords Spectral flow · Levinson's theorem · Schrödinger operators

Mathematics Subject Classification 47A40 · 58J30 · 81U20 · 81U24

1 Introduction

Much work has been done in recent years investigating the topological nature of Levinson's theorem from quantum scattering theory, both as an index theorem [3, 5, 24, 25, 33–35] and as an index pairing [1, 2, 4]. In this paper, we prove Levinson's theorem for Hamiltonians H_0, H on $\mathbb{R}^n$ by using spectral flow from H_0 to H. By applying the operator pseudodifferential calculus to the spectral flow formula of [15], we obtain a proof of the integral form of Levinson's theorem in all dimensions and in the presence of zero energy resonances. The dominant contribution is from the eta invariants of the endpoints H_0, H, and can be computed using the Birman–Kreĭn formula. In particular, we give a new approach to the relationship between the spectral shift function and spectral flow, extending the work of Azamov et al. [8, 9]. Our main result (see Theorem 4.2) is

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Theorem (Levinson's theorem) *Suppose that* $V \in C_c^\infty(\mathbb{R}^n)$*. Then the number* N *of eigenvalues* (*counted with multiplicity*) *of* $H = H_0 + V$ *is given by*

$$-N = \frac{1}{2\pi i}\int_0^\infty \big(\mathrm{Tr}(S(\lambda)^* S'(\lambda)) - p_n(\lambda)\big)\,\mathrm{d}\lambda - \beta_n(V) + N_{\mathrm{res}}$$

where N_{res} *is the contribution from resonances as defined in Theorem 2.15 and the polynomial* p_n *and constant* β_n *are defined in Lemma 2.10 and Definition 2.11.*

The layout of the paper is as follows. In Sect. 2.1 we recall the definition of spectral flow due to Phillips [30, 31] and the general formula for the spectral flow along a path of unbounded operators from [15]. In Sect. 2.2 we summarise the stationary scattering theory for the Hamiltonians H_0, H and in Sect. 2.3 we recall the spectral shift function and its defining properties, including the Birman–Kreĭn trace formula. In Sect. 2.4 we describe the high-energy behaviour of the spectral shift function from [1] and the pseudodifferential expansion of the resolvent from [14].

In Sect. 3 we use the scattering techniques of Sect. 2 to analyse the spectral flow formula in two components. The first is an 'integral of one-form' type term in Sect. 3.1 and the second is a Birman–Kreĭn contribution in Sect. 3.2. Finally, in Sect. 4 we obtain a formula for the spectral flow in terms of scattering data and as a consequence prove Levinson's theorem.

2 Background and notations

2.1 Spectral flow

The concept of spectral flow was used by Atiyah et al. in [6, 7] as a tool to develop APS index theory. Spectral flow is intuitively defined as the net number of eigenvalues which change sign along a path of self-adjoint operators, with the convention that an eigenvalue changing from negative to positive will provide a contribution of 1 to the spectral flow. We use the definition due to Phillips [30, 31]. Phillips' definition of spectral flow is valid in the much broader setting of semifinite von Neumann algebras with faithful normal semifinite traces, and while we do not need the full power of such a definition, we do need the ability to handle operators with continuous spectrum.

Consider the compact operators $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ with trace Tr and let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denote the projection onto the Calkin algebra. Let $\chi = \chi_{[0,\infty)}$ be the characteristic function of the interval $[0, \infty)$. Let $(T_t)_{t\in[0,1]}$ be any norm-continuous path of bounded self-adjoint Fredholm operators in $\mathcal{B}(\mathcal{H})$, so that $\pi(T_t)$ is a norm continuous path of invertibles. Then $\pi(\chi(T_t)) = \chi(\pi(T_t))$. Since the spectrum of the $\pi(T_t)$ are bounded away from zero, the path $\chi(\pi(T_t))$ is continuous. By compactness (and [11, Lemma 4.1]) we can choose a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ such that

$$\|\pi(\chi(T_t)) - \pi(\chi(T_s))\| < \frac{1}{2}$$

for all $t, s \in [t_{i-1}, t_i]$ and $1 \leq i \leq k$. Defining the projection $P_i = \chi\left(T_{t_i}\right)$ we find that $P_{i-1}P_i : P_i\mathcal{H} \to P_{i-1}\mathcal{H}$ is Fredholm. We recall the following definition, due to Phillips [30, 31].

Definition 2.1 Let $\mathcal{H}$ be a Hilbert space. For $t \in [0, 1]$ let (T_t) be any norm-continuous path of bounded self-adjoint Fredholm operators in $\mathcal{B}(\mathcal{H})$. For a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval $[0, 1]$ define the operators $P_i = \chi\left(T_{t_i}\right)$. Then we define the spectral flow of the path (T_t) by

$$\mathrm{sf}(T_t) := \sum_{i=1}^{k} \mathrm{Index}\,(P_{i-1}P_i)\,.$$

We note that the above definition of spectral flow is independent of the choice of partition [27, 30, 31] and agrees with the topological definition used in [6, 7] when both make sense. For unbounded operators, we make the following definition of spectral flow [13].

Definition 2.2 Let $\mathcal{H}$ be a Hilbert space with trace Tr . Let (D_t) be a graph norm continuous path of unbounded self-adjoint Fredholm operators on $\mathcal{H}$. Define the function $F : \mathbb{R} \to [-1, 1]$ by $F(x) = x(1 + x^2)^{-\frac{1}{2}}$. The spectral flow along the path (D_t) is defined by

$$\mathrm{sf}(D_t) := \mathrm{sf}(F(D_t)).$$

Throughout the rest of this section $[0, 1] \ni t \mapsto D_t$ stands for a path of unbounded self-adjoint linear Fredholm operators acting on some dense domain in $\mathcal{H} = L^2(\mathbb{R}^n)$. We denote by $(F_t) = (F(D_t))$ the bounded transform of the path (D_t). We must also impose a smoothness assumption on D_t to use analytic formulae for the spectral flow.

Definition 2.3 1. A path $[0, 1] \ni t \mapsto D_t$ is called Γ-differentiable at the point $t = t_0$ if and only if there exists a bounded linear operator T such that

$$\lim_{t \to t_0} \left\| t^{-1}(D_t - D_{t_0})(\mathrm{Id} + D_{t_0}^2)^{-\frac{1}{2}} - T \right\| = 0.$$

In this case we set $\dot{D}_{t_0} = T(\mathrm{Id} + D_{t_0}^2)^{\frac{1}{2}}$. The operator $\dot{D}_t$ is a symmetric linear operator with domain $\mathrm{Dom}(D_t)$ [15, Lemma 25].

2. If the mapping $t \mapsto \dot{D}_t(\mathrm{Id} + D_t^2)^{-\frac{1}{2}}$ is defined and continuous with respect to the operator norm, then we call the path $t \mapsto D_t$ a continuously Γ-differentiable or a C^1_Γ path.

The most general analytic spectral flow formula for the case of unbounded operators on a Hilbert space is given by the following theorem [15, Theorem 9]. The sign of the second term in (2.1) below appears incorrectly in [15, Theorem 9].

Theorem 2.4 *Let* $[0, 1] \ni t \mapsto D_t$ *be a piecewise* C^1_Γ *path of linear operators and* $F_t \in \mathcal{B}(\mathcal{H})$ *be Fredholm with* $\|F_t\| \leq 1$. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a positive* C^2 *function such that*

1. $\int_{\mathbb{R}} g(x)\,\mathrm{d}x = 1$;
2. $\int_0^1 \left\| \dot{D}_t g(D_t) \right\|_1 \mathrm{d}t < \infty$; *and*
3. $G(D_1) - \frac{1}{2}B_1 - G(D_0) + \frac{1}{2}B_0 \in \mathcal{L}^1(\mathcal{H})$, *where* $B_j = 2\chi_{[0,\infty)}(D_j) - 1$, *and* G *is the antiderivative of* g *such that* $G(\pm\infty) = \pm\frac{1}{2}$.

Then

$$\mathrm{sf}(D_t) = \int_0^1 \mathrm{Tr}\left(\dot{D}_t g(D_t)\right) \mathrm{d}t - \mathrm{Tr}\left(G(D_1) - \frac{1}{2}B_1 - G(D_0) + \frac{1}{2}B_0\right). \quad (2.1)$$

In our applications of this formula we will take $D_t = H_0 + \alpha\mathrm{Id} + tV$ where H_0 is the free Hamiltonian, α a carefully chosen constant and V a suitable potential. We now describe these ingredients.

2.2 Stationary scattering theory

We consider the scattering theory on $\mathbb{R}^n$ associated to the operators

$$H_0 = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} = -\Delta \quad \text{and} \quad H = H_0 + V,$$

where the (multiplication operator by the) potential V is a smooth compactly supported and real-valued function. With $\langle\cdot,\cdot\rangle$ the Euclidean inner product on $\mathbb{R}^n$, we denote the Fourier transform by

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \qquad [\mathcal{F}f](\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-i\langle x,\xi\rangle} f(x)\,\mathrm{d}x.$$

Note that the Fourier transform $\mathcal{F}$ is an isomorphism from $H^{s,t}$ to $H^{t,s}$ for any $s, t \in \mathbb{R}$.

We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{L}^1(\mathcal{H}_1, \mathcal{H}_2)$ the bounded, compact and trace class operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For $z \in \mathbb{C}\backslash\mathbb{R}$, we let

$$R_0(z) = (H_0 - z)^{-1}, \quad R(z) = (H - z)^{-1}.$$

The operator H_0 has purely absolutely continuous spectrum, and in particular no kernel. The operator H can have finitely many eigenvalues which are negative, or zero [36, Theorem 6.1.1].

Several Hilbert spaces recur, and we adopt the notation (following [21, Section 2] which contains an excellent discussion on the relations between the spaces and operators we introduce here)

$$\mathcal{H} = L^2(\mathbb{R}^n), \quad \mathcal{P} = L^2(\mathbb{S}^{n-1}), \quad \mathcal{H}_{\mathrm{spec}} = L^2(\mathbb{R}^+, \mathcal{P}) \cong L^2(\mathbb{R}^+) \otimes \mathcal{P}.$$

Here $\mathcal{H}_{\mathrm{spec}}$ provides the Hilbert space on which we can diagonalise the free Hamiltonian H_0.

Since V is bounded, $H = H_0 + V$ is self-adjoint with $\mathrm{Dom}(H) = \mathrm{Dom}(H_0)$. Since $V \in C_c^\infty(\mathbb{R}^n)$, the wave operators

$$W_\pm = \underset{t\to\pm\infty}{\text{s-lim}}\, \mathrm{e}^{itH}\mathrm{e}^{-itH_0}$$

exist and are asymptotically complete [36, Theorem 1.6.2]. The wave operators are partial isometries satisfying $W_\pm^* W_\pm = \mathrm{Id}$ and $W_\pm W_\pm^* = P_{\mathrm{ac}}$, the projection onto the absolutely continuous subspace for H. The scattering operator is the unitary operator

$$S = W_+^* W_-, \tag{2.2}$$

which commutes strongly with the free Hamiltonian H_0. For our analysis of the scattering operator, we describe the explicit unitary which diagonalises the free Hamiltonian.

Definition 2.5 Define the operator which diagonalises the free Hamiltonian H_0 as

$$F_0 : \mathcal{H} \to \mathcal{H}_{\mathrm{spec}} \quad \text{by} \quad [F_0 f](\lambda, \omega) = 2^{-\frac{1}{2}}\lambda^{\frac{n-2}{4}}[\mathcal{F} f](\lambda^{\frac{1}{2}}\omega).$$

By [21, p. 439] the operator F_0 is unitary and for $\lambda \in [0, \infty)$, $\omega \in \mathbb{S}^{n-1}$ and $f \in \mathcal{H}_{\mathrm{spec}}$ we have

$$[F_0 H_0 F_0^* f](\lambda, \omega) = \lambda f(\lambda, \omega) =: \lambda f(\lambda, \omega).$$

As a consequence of the relation $SH_0 = H_0 S$, there exists a family $\{S(\lambda)\}_{\lambda\in\mathbb{R}^+}$ of unitary operators on $\mathcal{P} = L^2(\mathbb{S}^{n-1})$ such that for all $\lambda \in \mathbb{R}^+$, $\omega \in \mathbb{S}^{n-1}$ and $f \in \mathcal{H}$ we have

$$[F_0 S f](\lambda, \omega) = S(\lambda)[F_0 f](\lambda, \omega).$$

For historical reasons, we refer to $S(\lambda)$ as the scattering matrix at energy $\lambda \in \mathbb{R}^+$ since in dimension $n = 1$ the operator $S(\lambda)$ is an $M_2(\mathbb{C})$-valued function.

Note that the operators H_0 and H are not Fredholm, since 0 is in the essential spectrum of both. To use the spectral flow formula of Theorem 2.4 we make the following adjustment for the rest of this article. Let $\nu \leq 0$ be the furthest eigenvalue of H from zero. We fix $\alpha > -2\nu + 1$, so that the operators $H_0(\alpha) = H_0 + \alpha$ and $H(\alpha) = H + \alpha$ define Fredholm operators. As a consequence, the path

$$[0, 1] \ni t \mapsto H_0 + tV + \alpha =: H_t(\alpha)$$

defines a C_Γ^1 path of Fredholm operators with $\dot{H}_t(\alpha) = V$. The operator $H_0(\alpha)$ has purely absolutely continuous spectrum $\sigma(H_0(\alpha)) = [\alpha, \infty)$ and the operator $H(\alpha)$ has absolutely continuous spectrum $\sigma_{\mathrm{ac}}(H(\alpha)) = \sigma(H_0(\alpha))$. In addition, the operator $H(\alpha)$ has a finite number of distinct eigenvalues $0 < \lambda_1(\alpha) < \lambda_2(\alpha) < \cdots < \lambda_K(\alpha) \leq \alpha$ of finite multiplicity. The eigenvalues satisfy $\lambda_j(\alpha) = \lambda_j + \alpha$, with $\lambda_1 < \lambda_2 < \cdots < \lambda_K \leq 0$ the distinct eigenvalues of H. We write $M(\lambda_j) = M(\lambda_j(\alpha))$

for the multiplicity of the eigenvalue λ_j and use the notation N_0 for the multiplicity of the zero eigenvalue for H. We also write

$$N = \sum_{j=1}^{K} M(\lambda_j)$$

for the total number of eigenvalues of H (counted with multiplicity). Let $P_{\mathrm{ac}}(H_0(\alpha))$ denote the projection onto the absolutely continuous spectrum for $H_0(\alpha)$. The wave operators

$$W_\pm(\alpha) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH(\alpha)}\mathrm{e}^{-itH_0(\alpha)} P_{\mathrm{ac}}(H_0(\alpha)) = W_\pm$$

exist and are asymptotically complete by the invariance principle [32, Theorem XI.11]. Direct calculation gives the following diagonalisation for $H_0(\alpha)$.

Lemma 2.6 *The operator* $F_\alpha : \mathcal{H} \to L^2([\alpha,\infty)) \otimes L^2(\mathbb{S}^{n-1})$ *given by*

$$[F_\alpha f](\lambda,\omega) = [F_0 f](\lambda-\alpha,\omega)$$

satisfies

$$[F_\alpha H_0(\alpha) f](\lambda,\omega) = \lambda [F_\alpha f](\lambda,\omega).$$

The scattering operator $S = W_+^* W_-$ is unitary and commutes with $H_0(\alpha)$ and so there exists a family $\{S_\alpha(\lambda)\}_{\lambda\in[\alpha,\infty)}$ of unitary operators on $L^2(\mathbb{S}^{n-1})$ such that

$$[F_\alpha S f](\lambda,\omega) = S_\alpha(\lambda)[F_\alpha f](\lambda,\omega).$$

In fact, we have $S_\alpha(\lambda) = S(\lambda-\alpha)$ for all $\lambda \in [\alpha,\infty)$. Pointwise we have $S_\alpha(\lambda) - \mathrm{Id} \in \mathcal{L}^1(L^2(\mathbb{S}^{n-1}))$, [36, Proposition 8.1.5].

2.3 The spectral shift function and the Birman–Kren trace formula

We now recall the spectral shift function [12, 26] for the pair $(H(\alpha), H_0(\alpha))$ and some of its defining properties (see [36, Theorems 0.9.2 and 0.9.7]). The proofs in [36] only consider $\alpha = 0$, however extend directly to $\alpha > 0$ by translation.

Theorem 2.7 *Suppose that* $V \in C_c^\infty(\mathbb{R}^n)$, $\alpha \geq 0$ *and let* S *be the corresponding scattering operator. Then there exists a unique (up to an additive constant) real-valued piecewise-*C^1 *function* $\xi_\alpha(\cdot, H, H_0) : \mathbb{R} \to \mathbb{R}$ *such that*

$$\mathrm{Tr}(f(H(\alpha)) - f(H_0(\alpha))) = \int_{\mathbb{R}} \xi_\alpha(\lambda, H, H_0) f'(\lambda)\,\mathrm{d}\lambda, \tag{2.3}$$

at least for all $f \in C^2(\mathbb{R})$ *with two locally bounded derivatives and satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\lambda^{m+1} f'(\lambda)\right) = O(\lambda^{-1-\varepsilon}) \tag{2.4}$$

as $\lambda \to \infty$, *for some* $\varepsilon > 0$ *and* $m > \frac{n}{2}$. *We specify* $\xi_\alpha(\cdot, H, H_0)$ *uniquely by the convention* $\xi_\alpha(\lambda, H, H_0) = 0$ *for* λ *sufficiently negative. Thus for* $\lambda < \alpha$, $\xi_\alpha(\cdot, H, H_0)$ *satisfies the relation*

$$\xi_\alpha(\lambda, H, H_0) = -\sum_{k=1}^{K} M(\lambda_k(\alpha))\, \chi_{[\lambda_k(\alpha),\infty)}(\lambda),$$

where we have indexed the distinct eigenvalues of $H(\alpha)$ *as* $\lambda_1(\alpha) < \cdots < \lambda_K(\alpha)$ *and each* $\lambda_j(\alpha)$ *has multiplicity* $M(\lambda_j(\alpha))$. *Furthermore, we have* $\xi_\alpha(\cdot, H, H_0)|_{(\alpha,\infty)} \in C^1(\alpha, \infty)$ *and for* $\lambda > \alpha$ *the relations*

$$\mathrm{Det}(S_\alpha(\lambda)) = \mathrm{e}^{-2\pi i \xi_\alpha(\lambda)} \quad \textit{and} \quad \mathrm{Tr}\left(S_\alpha(\lambda)^* S_\alpha'(\lambda)\right) = -2\pi i \xi_\alpha'(\lambda)$$

hold. Furthermore, we have $\xi_\alpha(\lambda) = \xi_0(\lambda - \alpha)$ *for almost all* $\lambda \in \mathbb{R}$.

We call $\xi_\alpha(\cdot, H, H_0)$ the spectral shift function for the pair $(H(\alpha), H_0(\alpha))$ and will often just write $\xi_\alpha = \xi_\alpha(\cdot, H, H_0)$. We also write $\xi = \xi_0$. Using integration by parts we can rewrite the defining property (2.3) in a sometimes more convenient fashion, known as the Birman–Kreĭn trace formula [18, Theorem III.4].

Lemma 2.8 *Suppose that* $V \in C_c^\infty(\mathbb{R}^n)$, $\alpha \geq 0$ *and let* S_α, ξ_α *be the corresponding scattering operator and spectral shift function. Then for all* $f \in C_c^\infty(\mathbb{R})$ *we have*

$$\mathrm{Tr}(f(H(\alpha)) - f(H_0(\alpha))) = \frac{1}{2\pi i}\int_\alpha^\infty f(\lambda)\mathrm{Tr}\left(S_\alpha(\lambda)^* S_\alpha'(\lambda)\right)\mathrm{d}\lambda + \sum_{k=1}^{K} f(\lambda_k(\alpha))M(\lambda_k(\alpha)) + f(\alpha)\left(\xi_\alpha(\alpha-) - \xi_\alpha(\alpha+) - M(\alpha)\right),$$

where we have defined $\xi_\alpha(\alpha\pm) = \lim_{\varepsilon\to 0^+} \xi_\alpha(\alpha \pm \varepsilon)$.

In fact by Theorem 2.7 we have, with N the total number of eigenvalues of H counted with multiplicity and $N_0 = M(\alpha)$ the number of zero eigenvalues for H, the relation $\xi_\alpha(\alpha-) = -N + N_0$. We can then rewrite the Birman–Kreĭn trace formula as

$$\mathrm{Tr}(f(H(\alpha)) - f(H_0(\alpha))) = \frac{1}{2\pi i}\int_\alpha^\infty f(\lambda)\mathrm{Tr}\left(S_\alpha(\lambda)^* S_\alpha'(\lambda)\right)\mathrm{d}\lambda + \sum_{k=1}^{K} f(\lambda_k(\alpha))M(\lambda_k(\alpha)) + f(\alpha)\left(-N - \xi_\alpha(\alpha+)\right).$$

2.4 Resolvent expansions and limiting behaviour of the spectral shift function

For $k \in \mathbb{N} \cup \{0\}$ and $f \in C_c^\infty(\mathbb{R}^n)$ we introduce the notation $f^{(k)} = [H_0, [H_0, [\cdots, [H_0, f]\cdots]]]$, where the expression has k commutators. We recall the following pseudodifferential expansion of the resolvent [1, Lemma 4.8] (see also [14, Lemma 6.11]).

Lemma 2.9 *Suppose that $V \in C_c^\infty(\mathbb{R}^n)$. For all $M, K \geq 0$ and $z \notin \sigma(H)$ we have the expansion*

$$R(z) = (H - z)^{-1} = \sum_{m=0}^{M} \left(\sum_{|k|=0}^{K} C_m(k)(-1)^{m+|k|} V^{(k_1)} \cdots V^{(k_m)} R_0(z)^{m+|k|+1} + P_{m,K}(z) \right) + (-1)^{M+1}(R_0(z)V)^{M+1} R(z),$$

where the remainder $P_{m,K}(z)$ is a pseudodifferential operator of order at most $-2m - K - 3$. The combinatorial coefficients $C_m(k)$ are given by

$$C_m(k) = \frac{(m + |k|)!}{k_1! \cdots k_m!(k_1 + 1)(k_1 + k_2 + 2) \cdots (|k| + m)}.$$

Note that the operator $V^{(k_1)} \cdots V^{(k_m)}$ is a differential operator of order at most $|k|$ with smooth compactly supported coefficients and thus we may write

$$V^{(k_1)} \cdots V^{(k_m)} = \sum_{|\beta|=0}^{|k|} g_{k,\beta} \partial^\beta, \tag{2.5}$$

where the multi-indices β are of length n and $g_{k,\beta} \in C_c^\infty(\mathbb{R}^n)$.

We now recall the high-energy behaviour of the spectral shift function and its derivative [1, Lemma 2.15, Theorem 4.15 and Remark 4.16].

Lemma 2.10 *Suppose that $V \in C_c^\infty(\mathbb{R}^n)$. Then for $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$ there exist coefficients $C_\ell(n, V), c_\ell(n, V), \beta_n(V)$ such that*

$$0 = \lim_{\lambda\to\infty} \left(-2\pi i \xi(\lambda) - 2\pi i \beta_n(V) - \sum_{\ell=1}^{\lfloor \frac{n-1}{2} \rfloor} C_\ell(n, V)\lambda^{\frac{n}{2}-\ell} \right) = \lim_{\lambda\to\infty} \left(-2\pi i \xi'(\lambda) - \sum_{\ell=1}^{\lfloor \frac{n-1}{2} \rfloor} c_\ell(n, V)\lambda^{\frac{n}{2}-\ell-1} \right).$$

The coefficients are related by $c_\ell(n,V) = \left(\frac{n}{2}-\ell\right) C_\ell(n,V)$. *For* $1 \le \ell \le \lfloor\frac{n-1}{2}\rfloor$ *and* $M, K \in \mathbb{N}$ *with* $M + K \ge n$ *we define the set*

$$Q_{M,K}(\ell) = \Big\{(m,k,\beta) \in \{0,1,\dots,M\} \times \{0,1,\dots,K\}^m \times \{0,1,\dots,K\}^n : |\beta| \le |k|, \text{ and } m + |k| + 1 - \frac{|\beta|}{2} = \ell\Big\}.$$

The coefficients $C_\ell(n,V)$ *are given by*

$$C_\ell(n,V) = \sum_{(m,k,\beta)\in Q_{M,K}(j)} \frac{(-1)^{m+|k|+1}(2\pi i)C_m(k)(-i)^{|\beta|}\Gamma\left(\frac{\beta_1+1}{2}\right)\cdots\Gamma\left(\frac{\beta_n+1}{2}\right)}{(m+1)(m+|k|)!\Gamma\left(\frac{n}{2}-m-|k|+\frac{|\beta|}{2}\right)(2\pi)^n}\int_{\mathbb{R}^n} V(x) g_{k,\beta}(x)\,\mathrm{d}x, \tag{2.6}$$

and

$$\beta_n(V) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \frac{1}{2\pi i} C_{\frac{n}{2}}(n,V), & \text{if } n \text{ is even}. \end{cases} \tag{2.7}$$

Definition 2.11 Define the functions $P_n, p_n : (0,\infty) \to \mathbb{C}$ by

$$P_n(\lambda) = 2\pi i \beta_n(V) + \sum_{\ell=1}^{\lfloor\frac{n-1}{2}\rfloor} C_\ell(n,V)\lambda^{\frac{n}{2}-\ell},$$

$$p_n(\lambda) = \sum_{\ell=1}^{\lfloor\frac{n-1}{2}\rfloor} c_\ell(n,V)\lambda^{\frac{n}{2}-\ell-1} = P_n'(\lambda).$$

We call P_n the high-energy polynomial for ξ and p_n the high-energy polynomial for ξ'.

Remark 2.12 Recall the spectral shift functions ξ, ξ_α for the pairs (H, H_0) and $(H(\alpha), H_0(\alpha))$. Since $\xi_\alpha(\lambda) = \xi(\lambda - \alpha)$ for all almost all $\lambda \in \mathbb{R}$ we have that the high-energy polynomial for ξ_α is $P_n(\cdot - \alpha)$ and likewise for ξ' and $p_n(\cdot - \alpha)$.

We can explicitly compute the lowest order polynomials (see [10, 16]), finding $P_1 = 0$, and

$$P_2(\lambda) = -\frac{(2\pi i)\,\mathrm{Vol}(\mathbb{S}^1)}{2(2\pi)^2}\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x = -\frac{2\pi i}{4\pi}\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x,$$

$$P_3(\lambda) = -\frac{(2\pi i)\lambda^{\frac{1}{2}}\mathrm{Vol}(\mathbb{S}^2)}{2(2\pi)^3}\int_{\mathbb{R}^3} V(x)\,\mathrm{d}x = -\frac{(2\pi i)\lambda^{\frac{1}{2}}}{4\pi^2}\int_{\mathbb{R}^3} V(x)\,\mathrm{d}x,$$

$$P_4(\lambda) = -\frac{(2\pi i)\lambda\mathrm{Vol}(\mathbb{S}^3)}{2(2\pi)^4}\int_{\mathbb{R}^4} V(x)\,\mathrm{d}x + \frac{(2\pi i)\mathrm{Vol}(\mathbb{S}^3)}{4(2\pi)^4}\int_{\mathbb{R}^4} V(x)^2\,\mathrm{d}x.$$

The integrability properties of the derivative of the spectral shift function on $\mathbb{R}^+$ are well-known, see [22, Theorem 5.2] and [1, Lemma 4.12].

Lemma 2.13 *Suppose that* $V \in C_c^\infty(\mathbb{R}^n)$. *Then the function* $\mathrm{Tr}\left(S(\cdot)^* S'(\cdot)\right) - p_n$ *is integrable on* $\mathbb{R}^+$. *In particular, if* $n = 1, 2$ *we have* $\mathrm{Tr}\left(S(\cdot)^* S'(\cdot)\right) \in L^1(\mathbb{R}^+)$.

We now define zero-energy resonances, a low-energy phenomena known to provide obstructions to generic behaviour in scattering theory in low dimensions.

Definition 2.14 Suppose that $V \in C_c^\infty(\mathbb{R}^n)$. If $n \neq 2$ we say there is a resonance if there exists a non-zero bounded distributional solution to $H\psi = 0$. If $n = 2$ we say there is a p-resonance if there exists a non-zero distributional solution ψ to $H\psi = 0$ with $\psi \in L^q(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for some $q > 2$. We say that there is an s-resonance if there exists a non-zero bounded distributional solution ψ to $H\psi = 0$ with $\psi \notin L^q(\mathbb{R}^2)$ for all $q < \infty$.

General bounds on the resolvent of H [20] show that there can be no resonances for dimension $n \geq 5$.

We now recall the value of the spectral shift function at zero in all dimensions from [1, Corollary 5.11].

Theorem 2.15 *Suppose* $V \in C_c^\infty(\mathbb{R}^n)$. *Then the value of the spectral shift function at zero is given by* $\xi(0+) = -N - N_{\mathrm{res}}$, *where* $N_{\mathrm{res}} = 0$ *unless*

$$N_{\mathrm{res}} = \begin{cases} \frac{1}{2}, & \text{if } n = 1 \text{ and there are no resonances},\\ N_p, & \text{if } n = 2 \text{ and there are } N_p = 0, 1, 2 \ p\text{-resonances},\\ \frac{1}{2}, & \text{if } n = 3 \text{ and there are resonances},\\ 1, & \text{if } n = 4 \text{ and there are resonances}. \end{cases}$$

We note that the proof of Theorem 2.15 in [1] is as a corollary of Levinson's theorem, however, the result can be obtained directly using perturbation determinant methods in odd dimensions (see [28, 29] and [18, Theorem 3.3]).

3 Spectral flow for Schrödinger operators

In this section, we analyse the spectral flow formula of Theorem 2.4 applied to the path $H_t(\alpha)$ by making a particular choice of the function g and then taking residues.

Define for $\mathrm{Re}(s) > \frac{1}{2}$ the constants $C_s = \int_{\mathbb{R}} (1 + u^2)^{-s}\, \mathrm{d}u$. The functions $s \mapsto C_s$ have a pole at $s = \frac{1}{2}$ with residue equal to one. For $\mathrm{Re}(s) > \frac{1}{2}$ we define the eta function $\eta_s : \mathbb{R} \to \mathbb{C}$ by

$$\eta_s(x) = \frac{1}{C_s} \int_1^\infty x(1 + wx^2)^{-s} w^{-\frac{1}{2}}\, \mathrm{d}w = \frac{2}{C_s} \int_x^\infty (1 + v^2)^{-s}\, \mathrm{d}s,$$

where the second expression is valid only for $x > 0$. We can now use the function η_s to obtain a useful form of Theorem 2.4.

Lemma 3.1 *Let* $[0,1] \ni t \mapsto D_t$ *be a piecewise* C^1_Γ *path of linear operators with* $\dot{D}_t(1+D_t^2)^{-s}$ *trace-class for all* $s > \frac{n}{4}$. *Then*

$$\mathrm{sf}(D_t) = \operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^1 \mathrm{Tr}\left(\dot{D}_t(\mathrm{Id}+D_t^2)^{-s}\right)\mathrm{d}t + \frac{C_s}{2}\mathrm{Tr}\left(\eta_s(D_1)-\eta_s(D_0)+P_{\mathrm{Ker}(D_1)}-P_{\mathrm{Ker}(D_0)}\right)\right). \tag{3.1}$$

Proof For $s > \frac{n}{4}$, let $g_s : \mathbb{R} \to \mathbb{R}$ be given by $g_s(x) = C_s^{-1}(1+x^2)^{-s}$. Note that the antiderivative G_s of g_s with $G(\pm\infty) = \pm\frac{1}{2}$ is given by

$$G_s(x) = -\frac{1}{2} + \frac{1}{C_s}\int_{-\infty}^{x}(1+u^2)^{-s}\,\mathrm{d}u.$$

The function g_s is even and so G_s is odd. For $x > 0$ we have $G_s(x) = \frac{1}{2} - \frac{1}{2}\eta_s(x)$, while for $x < 0$ we have $G_s(x) = -\frac{1}{2} - \frac{1}{2}\eta_s(x)$. Thus applying Theorem 2.4 to g_s, G_s and multiplying both sides by C_s yields

$$C_s\mathrm{sf}(D_t) = \int_0^1 \mathrm{Tr}\left(\dot{D}_t(\mathrm{Id}+D_t^2)^{-s}\right)\mathrm{d}t + \frac{C_s}{2}\mathrm{Tr}\left(\eta_s(D_1)-\eta_s(D_0)+P_{\mathrm{Ker}(D_1)}-P_{\mathrm{Ker}(D_0)}\right). \tag{3.2}$$

The left-hand side of Eq. (3.2) is a meromorphic function of s with a simple pole at $s = \frac{1}{2}$ and thus defines a meromorphic continuation of the right-hand side of Eq. (3.2) with a simple pole at $s = \frac{1}{2}$. As a result, taking the residue at $s = \frac{1}{2}$ gives Eq. (3.1). □

Equation (3.1) is the starting point for our analysis of the spectral flow along the path $H_t(\alpha)$. There are two separate types of terms to be considered. The first is the "integral of one-form" term which is evaluated in Sect. 3.1 using the pseudodifferential expansion of Lemma 2.9 and the second is the η contribution which is evaluated in Sect. 3.2 using the Birman–Kreĭn trace formula.

3.1 The "integral of one form" term

We use the pseudodifferential expansion of Lemma 2.9 to compute an expansion for the integral of one form term in Theorem 2.4. After a fixed number of terms (depending on the dimension n) the remainder term will be holomorphic at $s = \frac{1}{2}$ and can be discarded. We begin with a residue computation.

Lemma 3.2 *For $\ell \in \mathbb{N}$, $\alpha > 0$ we have*

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^\infty u^{\ell-1}(1+(u+\alpha)^2)^{-s}\,\mathrm{d}u\right) = \sum_{\substack{j=0\\ j\ even}}^{\ell-1}\binom{\ell-1}{j}\frac{(-1)^{\ell-\frac{j}{2}-1}\alpha^{\ell-j-1}\Gamma\left(\frac{j+1}{2}\right)}{4\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}.$$

Proof Fix $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{\ell+1}{2}$. We make the substitution $v = u+\alpha$ and apply the binomial expansion to obtain

$$\begin{aligned}\int_0^\infty u^{\ell-1}(1+(u+\alpha)^2)^{-s}\,\mathrm{d}u &= \int_\alpha^\infty (v-\alpha)^{\ell-1}(1+v^2)^{-s}\,\mathrm{d}v\\ &= \sum_{j=0}^{\ell-1}\binom{\ell-1}{j}(-\alpha)^{\ell-j-1}\int_\alpha^\infty v^j(1+v^2)^{-s}\,\mathrm{d}v\\ &= \sum_{j=0}^{\ell-1}\binom{\ell-1}{j}(-\alpha)^{\ell-j-1}\int_0^\infty v^j(1+v^2)^{-s}\,\mathrm{d}v\\ &\quad -\sum_{j=0}^{\ell-1}\binom{\ell-1}{j}(-\alpha)^{\ell-j-1}\int_0^\alpha v^j(1+v^2)^{-s}\,\mathrm{d}v.\end{aligned}$$

Since the integrals from 0 to α are over a finite region, they are holomorphic at $s = \frac{1}{2}$ and thus have vanishing residue. So we compute for $0 \le j \le \ell-1$ that

$$\int_0^\infty v^j(1+v^2)^{-s}\,\mathrm{d}v = \frac{1}{2}\int_0^\infty w^{\frac{j+1}{2}-1}(1+w)^{-s}\,\mathrm{d}w = \frac{\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(s-\frac{j+1}{2}\right)}{2\Gamma(s)}.$$

Taking the residue at $s = \frac{1}{2}$ we find

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^\infty v^j(1+v^2)^{-s}\,\mathrm{d}v\right) = \begin{cases}\frac{(-1)^{\frac{j}{2}}\frac{j+1}{2}\Gamma\left(\frac{j+1}{2}\right)}{2\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}, & \text{if } j \text{ is even},\\ 0, & \text{otherwise},\end{cases}$$

from which the result follows. □

To evaluate some further traces, we need to be able to integrate polynomials over $\mathbb{S}^{n-1}$. We use the following result [17].

Lemma 3.3 *Let β be a multi-index of length n and let $P_\beta : \mathbb{R}^n \to \mathbb{C}$ be given by $P_\beta(x) = x^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n}$. Then*

$$\int_{\mathbb{S}^{n-1}} P_\beta(\omega)\,\mathrm{d}\omega = \begin{cases}0, & \textit{if some } \beta_j \textit{ is odd},\\ \frac{2\Gamma\left(\frac{\beta_1+1}{2}\right)\cdots\Gamma\left(\frac{\beta_n+1}{2}\right)}{\Gamma\left(\frac{n+|\beta|}{2}\right)}, & \textit{if all } \beta_j \textit{ are even}.\end{cases}$$

We now contribute the residue of the contribution from the "integral of one form" term to the spectral flow.

Proposition 3.4 *Suppose that $V \in C_c^\infty(\mathbb{R}^n)$ and $\alpha > -2\nu$. Then for n odd we have*

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^1 \operatorname{Tr}\left(V\left(\mathrm{Id}+(H_0+tV+\alpha)^2\right)^{-s}\right)\mathrm{d}t\right)=0.$$

If n is even we have

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^1 \operatorname{Tr}\left(V\left(\mathrm{Id}+(H_0+tV+\alpha)^2\right)^{-s}\right)\mathrm{d}t\right)$$
$$=\sum_{\ell=1}^{\frac{n}{2}}\sum_{\substack{j=0\\ j\text{ even}}}^{\frac{n}{2}-\ell}\binom{\frac{n}{2}-\ell}{j}\frac{(-1)^{\frac{n}{2}-\ell-\frac{j}{2}-1}\alpha^{\frac{n}{2}-\ell-j}\Gamma\left(\frac{j+1}{2}\right)}{4(2\pi i)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}C_\ell(n,V),$$

with the $C_\ell(n, V)$ the high-energy coefficients defined in Eq. (2.6).

Proof For $\alpha > 0$ and $\mathrm{Re}(s) > \frac{n}{4}$ we define the function $\varphi_{\alpha,s} : \mathbb{R} \to \mathbb{C}$ by

$$\varphi_{\alpha,s}(x)=\left(1+(x+\alpha)^2\right)^{-s},$$

using the principal branch of the logarithm. The function $\varphi_{\alpha,s}$ is holomorphic in the half-plane $\mathrm{Re}(z) > -\alpha$. Let $a \in \left(-\frac{\alpha}{2}, 0\right)$ so that $a < \lambda$ for all $t \in [0, 1]$ and $\lambda \in \sigma(H_0 + tV)$ and define the vertical line $\gamma = \{a + iv : v \in \mathbb{R}\}$. For $t \in [0, 1]$ we use Cauchy's integral formula to write

$$\varphi_{\alpha,s}(H_0+tV)=-\frac{1}{2\pi i}\int_\gamma \varphi_{\alpha,s}(z)(H_0+tV-z)^{-1}\,\mathrm{d}z. \tag{3.3}$$

Denoting $R_t(z) = (H_0 + tV - z)^{-1}$ we have by Lemma 2.9 that for all $M, K \geq 0$ that

$$R_t(z)=\sum_{m=0}^{M}\left(t^m\sum_{|k|=0}^{K}(-1)^{m+|k|}C_m(k)V^{(k_1)}\cdots V^{(k_m)}R_0(z)^{m+|k|+1}+P_{m,K,t}(z)\right)$$
$$+(-1)^{M+1}t^{M+1}(R_0(z)V)^{M+1}R_t(z),$$

where $P_{m,K,t}(z)$ has order (at most) $-2m-K-3$. We can now write Eq. (3.3) as

$$\begin{aligned}\varphi_{\alpha,s}(H_0+tV) &= -\frac{1}{2\pi i}\sum_{m=0}^{M}\Big(t^m\sum_{|k|=0}^{K}(-1)^{m+|k|}C_m(k)V^{(k_1)}\cdots V^{(k_m)}\int_\gamma \varphi_{\alpha,s}(z)R_0(z)^{m+|k|+1}\,\mathrm{d}z\\ &\quad+\int_\gamma \varphi_{\alpha,s}(z)P_{m,K,t}(z)\,\mathrm{d}z\Big)+\frac{(-1)^M t^{M+1}}{2\pi i}\int_\gamma \varphi_{\alpha,s}(z)(R_0(z)V)^{M+1}R_t(z)\,\mathrm{d}z\\ &:= -\frac{1}{2\pi i}\sum_{m=0}^{M}t^m\sum_{|k|=0}^{K}(-1)^{m+|k|}V^{(k_1)}\cdots V^{(k_m)}\int_\gamma \varphi_{\alpha,s}(z)R_0(z)^{m+|k|+1}\,\mathrm{d}z\\ &\quad+E(M,K,t,\alpha,s).\end{aligned}$$

Using again Cauchy's integral formula we can compute that

$$\frac{1}{2\pi i}\int_\gamma \varphi_{\alpha,s}(z)R_0(z)^{m+|k|+1}\,\mathrm{d}z = -\frac{1}{(m+|k|)!}\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}z^{m+|k|}}|_{z=H_0}.$$

Thus we have the expression

$$\begin{aligned}\varphi_{\alpha,s}(H_0+tV) &= \sum_{m=0}^{M}\sum_{|k|=0}^{K}\frac{C_m(k)(-1)^{m+|k|}t^m}{(m+|k|)!}V^{(k_1)}\\ &\cdots V^{(k_m)}\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}z^{m+|k|}}|_{z=H_0}+E(M,K,t,\alpha,s).\end{aligned}$$

Choose $M=\lfloor n\rfloor$ and for $0\le m\le M$ let $K=M-m$. Since $V^{(k_1)}\cdots V^{(k_m)}$ is a differential operator of order $|k|$, we can write

$$V^{(k_1)}\cdots V^{(k_m)} = \sum_{|\beta|=0}^{|k|} g_{k,\beta}\partial^\beta,$$

where β is a multi-index of length n and $g_{k,\beta} \in C_c^\infty(\mathbb{R}^n)$. Then we can use Lemma 3.3 to compute

$$\begin{aligned}
&\mathrm{Tr}\left(VV^{(k_1)}\cdots V^{(k_m)}\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}z^{m+|k|}}|_{z=H_0}\right) = \sum_{|\beta|=0}^{|k|}\mathrm{Tr}\left(Vg_{k,\beta}\partial^\beta\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}z^{m+|k|}}|_{z=H_0}\right)\\
&= (2\pi)^{-n}\sum_{|\beta|=0}^{|k|}\left(\int_{\mathbb{R}^n}V(x)g_{k,\beta}(x)\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^n}(-i)^{|\beta|}\xi^\beta\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}z^{m+|k|}}(|\xi|^2)\,\mathrm{d}\xi\right)\\
&= \sum_{\substack{|\beta|=0\\ \beta\text{ even}}}^{|k|}\frac{2(-i)^{|\beta|}\Gamma\left(\frac{\beta_1+1}{2}\right)\cdots\Gamma\left(\frac{\beta_n+1}{2}\right)}{(2\pi)^n\Gamma\left(\frac{n+|\beta|}{2}\right)}\left(\int_{\mathbb{R}^n}V(x)g_{k,\beta}(x)\mathrm{d}x\right)\\
&\quad\times\left(\int_0^\infty r^{n+|\beta|-1}\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}z^{m+|k|}}(r^2)\,\mathrm{d}r\right)\\
&= \sum_{\substack{|\beta|=0\\ \beta\text{ even}}}^{|k|}\frac{(-i)^{|\beta|}\Gamma\left(\frac{\beta_1+1}{2}\right)\cdots\Gamma\left(\frac{\beta_n+1}{2}\right)}{(2\pi)^n\Gamma\left(\frac{n+|\beta|}{2}\right)}\left(\int_{\mathbb{R}^n}V(x)g_{k,\beta}(x)\,\mathrm{d}x\right)\\
&\quad\times\left(\int_0^\infty u^{\frac{n+|\beta|}{2}-1}\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}u^{m+|k|}}(u)\,\mathrm{d}u\right),
\end{aligned}$$

where the sum is over multi-indices β with all β_j even. First, suppose that $m+|k| \leq \frac{n+|\beta|}{2}-1$. Integrating by parts in the u integral $(m+|k|)$ times we find

$$\begin{aligned}
&\mathrm{Tr}\left(VV^{(k_1)}\cdots V^{(k_m)}\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}z^{m+|k|}}|_{z=H_0}\right)\\
&\quad= \sum_{\substack{|\beta|=0\\ \beta\text{ even}}}^{|k|}\frac{(-i)^{|\beta|}(-1)^{m+|k|}\Gamma\left(\frac{\beta_1+1}{2}\right)\cdots\Gamma\left(\frac{\beta_n+1}{2}\right)}{(2\pi)^n\Gamma\left(\frac{n+|\beta|}{2}\right)}\left(\int_{\mathbb{R}^n}V(x)g_{k,\beta}(x)\,\mathrm{d}x\right)\\
&\times\left(\int_0^\infty\frac{\mathrm{d}^{m+|k|}}{\mathrm{d}u^{m+|k|}}\left(u^{\frac{n+|\beta|}{2}-1}\right)\varphi_{\alpha,s}(u)\,\mathrm{d}u\right)\\
&\quad= \sum_{\substack{|\beta|=0\\ \beta\text{ even}}}^{|k|}\frac{(-i)^{|\beta|}(-1)^{m+|k|}\Gamma\left(\frac{\beta_1+1}{2}\right)\cdots\Gamma\left(\frac{\beta_n+1}{2}\right)}{(2\pi)^n\Gamma\left(\frac{n+|\beta|}{2}-m-|k|\right)}\left(\int_{\mathbb{R}^n}V(x)g_{k,\beta}(x)\,\mathrm{d}x\right)\\
&\times\left(\int_0^\infty u^{\frac{n+|\beta|}{2}-m-|k|-1}\varphi_{\alpha,s}(u)\,\mathrm{d}u\right).
\end{aligned}$$

Note that the boundary terms from the integration by parts vanish due to the inequality parts vanish since $\frac{n+|\beta|}{2}-m-|k|-1>0$. We first consider n odd and make the estimate

$$\left|\int_0^\infty u^{\frac{n+|\beta|}{2}-m-|k|-1}\varphi_{\alpha,s}(u)\,\mathrm{d}u\right| \le \int_0^\infty u^{\frac{n+|\beta|}{2}-m-|k|-1}(1+u^2)^{-\mathrm{Re}(s)}\,\mathrm{d}u$$
$$= \frac{1}{2}\int_0^\infty v^{\frac{n+|\beta|}{4}-\frac{m+|k|}{2}-1}(1+v)^{-\mathrm{Re}(s)}\,\mathrm{d}v$$
$$= \frac{\Gamma\left(\frac{n+|\beta|}{4}-\frac{m+|k|}{2}\right)\Gamma\left(\mathrm{Re}(s)+\frac{m+|k|}{2}-\frac{n+|\beta|}{4}\right)}{2\Gamma\left(\mathrm{Re}(s)\right)}.$$

Since n is odd, we have

$$\frac{1}{2}+\frac{m+|k|}{2}-\frac{n+|\beta|}{4}\notin\mathbb{Z}$$

for all possible m, k, β and thus we find

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^\infty u^{\frac{n+|\beta|}{2}-m-|k|-1}\varphi_{\alpha,s}(u)\,\mathrm{d}u\right)=0,$$

from which we deduce that

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^1 \mathrm{Tr}\left(V\left(\mathrm{Id}+(H_0+tV+\alpha)^2\right)^{-s}\right)\mathrm{d}t\right)=\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^1 \mathrm{Tr}\left(VE(M,K,t,\alpha,s)\ \mathrm{d}t\right)\right),$$

which we show is zero below. We now consider n even. An application of Lemma 3.2 gives

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^\infty (u^{\frac{n+|\beta|}{2}-m-|k|-1}\varphi_{\alpha,s}(u)\,\mathrm{d}u\right)$$
$$=\sum_{j=0}^{\frac{n+|\beta|}{2}-m-|k|-1}\binom{\frac{n+|\beta|}{2}-m-|k|-1}{j}\frac{(-1)^{\frac{n+|\beta|}{2}-m-|k|-1}\alpha^{\frac{n+|\beta|}{2}-m-|k|-j-1}\Gamma\left(\frac{j+1}{2}\right)}{4\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}.$$

Next, we consider the case $m+|k|>\frac{n+|\beta|}{2}-1$. If n is even, we integrate by parts $\frac{n+|\beta|}{2}-1$ times in the u-integral to obtain

$$\int_0^\infty u^{\frac{n}{2}-1}\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}u^{m+|k|}}(u)=(-1)^{\frac{n+|\beta|}{2}-1}\int_0^\infty\left(\frac{\mathrm{d}^{\frac{n+|\beta|}{2}-1}}{\mathrm{d}u^{\frac{n+|\beta|}{2}-1}}u^{\frac{n+|\beta|}{2}-1}\right)\left(\frac{\mathrm{d}^{m+|k|+1-\frac{n+|\beta|}{2}}\varphi_{\alpha,s}}{\mathrm{d}u^{m+|k|+1-\frac{n+|\beta|}{2}}}(u)\right)\mathrm{d}u$$
$$=(-1)^{\frac{n+|\beta|}{2}}\Gamma\left(\frac{n+|\beta|}{2}\right)\int_0^\infty\frac{\mathrm{d}^{m+|k|+1-\frac{n+|\beta|}{2}}\varphi_{\alpha,s}}{\mathrm{d}u^{m+|k|+1-\frac{n+|\beta|}{2}}}(u)\,\mathrm{d}u$$
$$=(-1)^{\frac{n+|\beta|}{2}+1}\Gamma\left(\frac{n+|\beta|}{2}\right)\frac{\mathrm{d}^{m+|k|-\frac{n+|\beta|}{2}}\varphi_{\alpha,s}}{\mathrm{d}u^{m+|k|+1-\frac{n+|\beta|}{2}}}\Big|_{u=0},$$

which is holomorphic at $s = \frac{1}{2}$. If n is odd, then a similar estimate to the case $m + |k| \le \frac{n+|\beta|}{2} - 1$ shows that the contribution is holomorphic at $s = \frac{1}{2}$. So we find

$$\begin{aligned}
&\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_0^1 t^m \mathrm{Tr}\left(V V^{(k_1)}\cdots V^{(k_m)}\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}z^{m+|k|}}|_{z=H_0}\right)\mathrm{d}t\right)\\
&= \sum_{\substack{|\beta|=0\\ \beta \text{ even}}}^{|k|}\ \sum_{\substack{j=0\\ j \text{ even}}}^{\frac{n+|\beta|}{2}-m-|k|-1}\left(\int_{\mathbb{R}^n} V(x) g_{k,\beta}(x)\,\mathrm{d}x\right)\binom{\frac{n+|\beta|}{2}-m-|k|-1}{j}\\
&\quad\times\frac{(-i)^{|\beta|}(-1)^{\frac{n+|\beta|}{2}-m-|k|-\frac{j+1}{2}}\alpha^{\frac{n+|\beta|}{2}-m-|k|-j-1}\Gamma\left(\frac{\beta_1+1}{2}\right)\cdots\Gamma\left(\frac{\beta_n+1}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}{(2\pi)^n(m+1)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+|\beta|}{2}-m-|k|\right)}.
\end{aligned}$$

For $\ell \in \mathbb{N}$ define the set

$$\begin{aligned}
Q_{M,K}(\ell) = \Big\{&(m,k,\beta)\in\{0,1,\dots,M\}\times\{0,1,\dots,K\}^m\times\{0,1,\dots,K\}^n : |\beta|\le|k|,\\
&m+|k|+1-\frac{|\beta|}{2}=\ell\Big\}.
\end{aligned}$$

Recalling the coefficients $C_\ell(n, V)$ of the high-energy polynomial for ξ of Eq. (2.6) we have that

$$\begin{aligned}
&\operatorname*{Res}_{s=\frac{1}{2}}\left(\sum_{m=0}^{M}\int_0^1 t^m\sum_{|k|=0}^{K}\frac{C_m(k)(-1)^{m+|k|}}{(m+|k|)!}\mathrm{Tr}\left(V V^{(k_1)}\cdots V^{(k_m)}\frac{\mathrm{d}^{m+|k|}\varphi_{\alpha,s}}{\mathrm{d}z^{m+|k|}}|_{z=H_0}\right)\mathrm{d}t\right)\\
&= \sum_{m=0}^{M}\sum_{|k|=0}^{K}\sum_{\substack{|\beta|=0\\ \beta \text{ even}}}^{|k|}\ \sum_{\substack{j=0\\ j \text{ even}}}^{\frac{n+|\beta|}{2}-m-|k|-1}\left(\int_{\mathbb{R}^n} V(x) g_{k,\beta}(x)\,\mathrm{d}x\right)\binom{\frac{n+|\beta|}{2}-m-|k|-1}{j}\\
&\quad\times\frac{(-i)^{|\beta|}(-1)^{\frac{n+|\beta|}{2}-m-|k|-\frac{j+1}{2}}\alpha^{\frac{n+|\beta|}{2}-m-|k|-j-1}\Gamma\left(\frac{\beta_1+1}{2}\right)\cdots\Gamma\left(\frac{\beta_n+1}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}{(2\pi)^n(m+1)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+|\beta|}{2}-m-|k|\right)}\\
&= \sum_{\ell=1}^{\frac{n}{2}}\ \sum_{\substack{j=0\\ j \text{ even}}}^{\frac{n}{2}-\ell}\binom{\frac{n}{2}-\ell}{j}\frac{(-1)^{\frac{n}{2}-\ell-\frac{j}{2}-1}\alpha^{\frac{n}{2}-\ell-j-1}\Gamma\left(\frac{j+1}{2}\right)}{4(2\pi i)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}C_\ell(n,V).
\end{aligned}$$

We now consider the contribution from the remainder term $E(M, K, t, \alpha, s)$. There are two types of terms to consider. The first are those involving $P_{m,K,t}(z)$. Since $V \in C_c^\infty(\mathbb{R}^n)$ we can factorise $V = q_1 q_2$ with $q_1, q_2 \in C_c^\infty(\mathbb{R}^n)$. Since $P_{m,K,t}(z)$

has order at most $-2m - K - 3$, there exists $C > 0$ such that

$$\left\| R_0(z)^{-m-\frac{K}{2}-\frac{3}{2}} q_2 P_{m,K,t}(z) \right\| \le C.$$

Note also that

$$\left\| q_1 R_0(z)^{m+\frac{K}{2}+\frac{3}{2}} \right\|_1 \le (a^2+v^2)^{-\frac{m}{2}-\frac{K}{4}-\frac{3}{4}+\frac{n+1}{4}},$$

which follows from a careful application of the Rellich lemma. Combining these we make the estimate

$$\left\| \int_\gamma \varphi_{\alpha,s}(z) P_{m,K,t}(z)\,\mathrm{d}z \right\|_1 \le C \int_{\mathbb{R}} (1+((a+\alpha)^2+v^2))^{-\frac{\mathrm{Re}(s)}{2}} (a^2+v^2)^{-\frac{m}{2}-\frac{K}{4}-\frac{3}{4}+\frac{n+1}{4}}\,\mathrm{d}v,$$

which is finite for $\mathrm{Re}(s) + m + \frac{K}{2} + \frac{3}{2} > \frac{n+1}{2} + 1$. Recalling that $\mathrm{Re}(s) > \frac{1}{2}$ and we have chosen $M = n$ and $K = M - m$ guarantees convergence. A similar argument to [14, Lemma 7.4] shows that this contribution is holomorphic at $s = \frac{1}{2}$, as is the contribution from the terms containing $R_t(z)$. Thus for both n even and odd, we find

$$\operatorname*{Res}_{s=\frac{1}{2}} \left(\int_0^1 \mathrm{Tr}\,(V E(M, K, t, \alpha, s)\ \mathrm{d}t \right) = 0,$$

which completes the proof. □

3.2 The Birman–Kreín term

In this subsection, we use the Birman–Kreĭn trace formula to determine the kernel and η contributions to the spectral flow.

Lemma 3.5 *By construction, the projections $P_{\mathrm{Ker}(H(\alpha))}$ and $P_{\mathrm{Ker}(H_0(\alpha))}$ are both zero.*

Since the kernel terms both vanish we are now able to evaluate the η contributions. We note that by Proposition 3.4 the residue of the integral of one form contribution to Eq. (3.1) at $s = \frac{1}{2}$ exists, and thus so does the residue of the Birman–Kreĭn contribution at $s = \frac{1}{2}$.

Lemma 3.6 *Suppose that $V \in C_c^\infty(\mathbb{R}^n)$. Then the η contribution to the Hamiltonian spectral flow is given by*

$$\begin{aligned}
&\operatorname*{Res}_{s=\frac{1}{2}} \Big(C_s \big(\mathrm{Tr}(\eta_s(H(\alpha)) - \eta_s(H_0(\alpha))\big)\Big) \\
&\quad = N + N_{\mathrm{res}} + \operatorname*{Res}_{s=\frac{1}{2}} \left(\frac{1}{2\pi i} \int_\alpha^\infty C_s \eta_s(\lambda) \mathrm{Tr}(S_\alpha(\lambda)^* S_\alpha'(\lambda))\,\mathrm{d}\lambda \right),
\end{aligned}$$

where N is the number of eigenvalues of $H = H_0 + V$, counted with multiplicity, and N_{res} is the contribution from resonances as defined in Theorem 2.15.

Proof Choose $s > \frac{n}{2} + 1$, so that Eq. (2.4) is satisfied for η_s and thus the Birman–Kreĭn trace formula can be applied to η_s. Enumerate the distinct eigenvalues of $H(\alpha)$ as $0 < \lambda_1(\alpha) < \cdots < \lambda_K(\alpha) \le \alpha$. We prove the result in the case $\lambda_K(\alpha) = \alpha$, that is in the case that zero is an eigenvalue for $H = H_0 + V$. Apply the Birman–Kreĭn trace formula to obtain

$$\begin{aligned}\mathrm{Tr}\,(\eta_s(H(\alpha)) - \eta_s(H_0(\alpha))) &= \frac{1}{2\pi i}\int_\alpha^\infty \eta_s(\lambda)\mathrm{Tr}(S_\alpha(\lambda)^* S'_\alpha(\lambda))\,\mathrm{d}\lambda \\ &\quad + \sum_{k=1}^{K-1} M(\lambda_k(\alpha))\eta_s(\lambda_k(\alpha)) \\ &\quad + M(\alpha)\eta_s(\alpha) + \eta_s(\alpha)\,(\xi_\alpha(\alpha-) - \xi_\alpha(\alpha+) - M(\alpha))\,,\end{aligned}$$

where ξ_α is the spectral shift function for the pair $(H(\alpha), H_0(\alpha))$ and $M(\lambda_j(\alpha))$ denotes the multiplicity of the eigenvalue $\lambda_j(\alpha)$ for the operator $H(\alpha)$. Recall that by construction, we have $\xi_\alpha(\lambda) = \xi(\lambda - \alpha)$ so that $\xi_\alpha(\alpha\pm) = \xi(0\pm)$. Thus after multiplying by C_s we have

$$\begin{aligned}C_s\mathrm{Tr}\,(\eta_s(H(\alpha)) - \eta_s(H_0(\alpha))) &= \frac{C_s}{2\pi i}\int_\alpha^\infty \eta_s(\lambda)\mathrm{Tr}(S_\alpha(\lambda)^* S'_\alpha(\lambda))\,\mathrm{d}\lambda \\ + C_s\sum_{k=1}^{K} M(\lambda_k(\alpha))\eta_s(\lambda_k(\alpha)) &+ C_s\eta_s(\alpha)\,(\xi(0-) - \xi(0+) - N_0)\,. \end{aligned}\tag{3.4}$$

Observe that for $x \neq 0$ we have $\mathrm{Res}_{s=\frac{1}{2}}\,(C_s\eta_s(x)) = \mathrm{sign}(x)$.

The left hand side of Eq. (3.4) has a residue at $s = \frac{1}{2}$ if and only if the first term on the right hand side does. Note that by Theorem 2.15 we have $\xi(0-) - \xi(0+) - N_0 = N_{\mathrm{res}}$. It remains to take the residue at $s = \frac{1}{2}$.

By construction we have $\lambda_j(\alpha) > 0$ for all j and thus

$$\begin{aligned}&\mathop{\mathrm{Res}}_{s=\frac{1}{2}}\,(C_s\mathrm{Tr}\,(\eta_s(H(\alpha)) - \eta_s(H_0(\alpha)))) \\ &\quad = \mathop{\mathrm{Res}}_{s=\frac{1}{2}}\left(\frac{C_s}{2\pi i}\int_\alpha^\infty \eta_s(\lambda)\mathrm{Tr}(S_\alpha(\lambda)^* S'_\alpha(\lambda))\,\mathrm{d}\lambda + \sum_{k=1}^{K} M(\lambda_k)C_s\eta_s(\lambda_k(\alpha)) + C_s\eta_s(\alpha)N_{\mathrm{res}}\right) \\ &\quad = N + N_{\mathrm{res}} + \mathop{\mathrm{Res}}_{s=\frac{1}{2}}\left(\frac{1}{2\pi i}\int_\alpha^\infty C_s\eta_s(\lambda)\mathrm{Tr}(S_\alpha(\lambda)^* S'_\alpha(\lambda))\,\mathrm{d}\lambda\right),\end{aligned}$$

as claimed. □

We can now compute the residue of the Birman–Kreĭn integral contribution to the spectral flow with the aid of a technical result.

Lemma 3.7 *Fix $\beta \geq 0$ and $f, g : \mathbb{R}^+ \to \mathbb{C}$ with $f - g \in L^1(\mathbb{R}^+)$. Suppose in addition that*

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(C_s \int_\beta^\infty \eta_s(\lambda) f(\lambda-\beta)\,\mathrm{d}\lambda\right)$$

exists. Then

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(C_s \int_\beta^\infty \eta_s(\lambda) f(\lambda-\beta)\,\mathrm{d}\lambda\right) = \int_0^\infty (f(\lambda) - g(\lambda))\,\mathrm{d}\lambda + \operatorname*{Res}_{s=\frac{1}{2}}\left(C_s \int_\beta^\infty \eta_s(\lambda) g(\lambda-\beta)\,\mathrm{d}\lambda\right).$$

Proof Adding zero gives

$$\begin{aligned}&\operatorname*{Res}_{s=\frac{1}{2}}\left(C_s \int_\beta^\infty \eta_s(\lambda) f(\lambda-\beta)\,\mathrm{d}\lambda\right)\\ &\quad= \operatorname*{Res}_{s=\frac{1}{2}}\left(C_s \int_\beta^\infty \eta_s(\lambda)(f(\lambda-\beta) - g(\lambda-\beta))\,\mathrm{d}\lambda + C_s \int_\beta^\infty \eta_s(\lambda) g(\lambda-\beta)\,\mathrm{d}\lambda\right).\end{aligned} \tag{3.5}$$

One straightforwardly checks that $\operatorname{Res}_{s=1/2} \eta_s(x) = 1$ for all $x > 0$. Then an application of the dominated convergence theorem allows us to compute that

$$\begin{aligned}\operatorname*{Res}_{s=\frac{1}{2}}\left(\int_\beta^\infty \eta_s(\lambda)(f(\lambda-\beta) - g(\lambda-\beta))\,\mathrm{d}\lambda\right) &= \int_\beta^\infty \operatorname*{Res}_{s=\frac{1}{2}} \eta_s(\lambda)(f(\lambda-\beta) - g(\lambda-\beta))\,\mathrm{d}\lambda\\ &= \int_\beta^\infty (f(\lambda-\beta) - g(\lambda-\beta))\,\mathrm{d}\lambda.\end{aligned}$$

Since the residue of the first term on the right-hand side of Eq. (3.5) exists, so does the residue of the second term on the right-hand side. Making the substitution $u = \lambda - \beta$ completes the proof. □

Proposition 3.8 *Suppose that $V \in C_c^\infty(\mathbb{R}^n)$ and $\alpha > -2\nu$. Then for n odd we have*

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\frac{C_s}{2\pi i} \int_\alpha^\infty \operatorname{Tr}(S_\alpha(\lambda)^* S_\alpha'(\lambda))\eta_s(\lambda)\,\mathrm{d}\lambda\right) = \frac{1}{2\pi i}\int_0^\infty \left(\operatorname{Tr}(S(\lambda)^* S'(\lambda)) - p_n(\lambda)\right)\mathrm{d}\lambda.$$

If n is even we have

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\frac{C_s}{2\pi i}\int_\alpha^\infty \mathrm{Tr}(S_\alpha(\lambda)^*S_\alpha'(\lambda))\eta_s(\lambda)\,\mathrm{d}\lambda\right)=\frac{1}{2\pi i}\int_0^\infty \big(\mathrm{Tr}(S(\lambda)^*S'(\lambda))-p_n(\lambda)\big)\,\mathrm{d}\lambda$$
$$+\sum_{\ell=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\substack{j=0\\ j\ even}}^{\frac{n}{2}-\ell}\binom{\frac{n}{2}-\ell}{j}\frac{(-1)^{\frac{n}{2}-\ell-\frac{j}{2}}\alpha^{\frac{n}{2}-\ell-j}\Gamma\left(\frac{j+1}{2}\right)}{2(2\pi i)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}C_\ell(n,V),$$

where the $C_\ell(n,V)$ are the high-energy coefficients for P_n defined in Eq. (2.6).

Proof Note that by Lemma 2.13 the map $[0,\infty)\ni\lambda\mapsto \mathrm{Tr}(S(\lambda)^*S'(\lambda))-p_n(\lambda)$ is integrable on $[0,\infty)$ and thus we can apply Lemma 3.7 to obtain

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\frac{C_s}{2\pi i}\int_\alpha^\infty \big(\mathrm{Tr}(S_\alpha(\lambda)^*S_\alpha'(\lambda))\big)\,\eta_s(\lambda)\,\mathrm{d}\lambda\right)$$
$$=\frac{1}{2\pi i}\int_0^\infty \big(\mathrm{Tr}(S(\lambda)^*S'(\lambda))-p_n(\lambda)\big)\,\mathrm{d}\lambda+\frac{1}{2\pi i}\operatorname*{Res}_{s=\frac{1}{2}}\left(C_s\int_\alpha^\infty p_n(\lambda-\alpha)\eta_s(\lambda)\,\mathrm{d}\lambda\right).$$

Thus it remains to compute

$$\operatorname*{Res}_{s=\frac{1}{2}}\left(\frac{C_s}{2\pi i}\int_\alpha^\infty p_n(\lambda-\alpha)\eta_s(\lambda)\,\mathrm{d}\lambda\right)=\operatorname*{Res}_{s=\frac{1}{2}}\left(\sum_{\ell=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{c_\ell(n,V)}{2\pi i}C_s\int_\alpha^\infty(\lambda-\alpha)^{\frac{n}{2}-\ell-1}\eta_s(\lambda)\,\mathrm{d}\lambda\right)$$
$$=\operatorname*{Res}_{s=\frac{1}{2}}\left(\sum_{\ell=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{c_\ell(n,V)}{2\pi i}C_s\int_0^\infty u^{\frac{n}{2}-\ell-1}\eta_s(u+\alpha)\,\mathrm{d}u\right).$$

We show that the residue of each of the terms in the sum exist individually, so that the summation can be passed through the residue. First we consider n odd. We integrate by parts to find

$$C_s\int_0^\infty u^{\frac{n}{2}-\ell-1}\eta_s(u+\alpha)\,\mathrm{d}u=\int_0^\infty\int_{u+\alpha}^\infty u^{\frac{n}{2}-\ell-1}(1+v^2)^{-s}\,\mathrm{d}v\,\mathrm{d}u$$
$$=\left[\frac{u^{\frac{n}{2}-\ell}}{\frac{n}{2}-\ell}\int_{u+\alpha}^\infty(1+v^2)^{-s}\,\mathrm{d}v\right]_0^\infty+\frac{1}{\frac{n}{2}-\ell}\int_0^\infty u^{\frac{n}{2}-\ell}(1+(u+\alpha)^2)^{-s}\,\mathrm{d}u$$
$$=\frac{1}{\frac{n}{2}-\ell}\int_0^\infty u^{\frac{n}{2}-\ell}(1+u^2)^{-s}\,\mathrm{d}u$$
$$+\frac{1}{\frac{n}{2}-\ell}\int_0^\infty u^{\frac{n}{2}-\ell}((1+(u+\alpha)^2)^{-s}-(1+u^2)^{-s}\,\mathrm{d}u$$
$$=\frac{\Gamma\left(\frac{n}{4}-\frac{\ell}{2}+\frac{1}{2}\right)\Gamma\left(s-\frac{n}{4}+\frac{\ell}{2}-\frac{1}{2}\right)}{2\Gamma(s)}+\mathrm{holo}(s)$$

where holo is a function holomorphic at $s = \frac{1}{2}$. Since n is odd, we have $1 - \frac{n}{4} + \frac{\ell}{2} \notin \mathbb{Z}$ and thus

$$\operatorname*{Res}_{s=\frac{1}{2}} \left(\frac{C_s}{2\pi i} \int_\alpha^\infty p_n(\lambda - \alpha)\eta_s(\lambda)\,\mathrm{d}\lambda \right) = \sum_{\ell=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{c_\ell(n, V)}{2\pi i} \operatorname*{Res}_{s=\frac{1}{2}} \left(C_s \int_0^\infty u^{\frac{n}{2}-\ell-1} \eta_s(u+\alpha)\,\mathrm{d}u \right)$$
$$= 0.$$

Now we consider n even. In this case we integrate by parts to write

$$\int_\alpha^\infty (\lambda - \alpha)^{\frac{n}{2}-\ell-1} \eta_s(\lambda)\,\mathrm{d}\lambda = -\frac{1}{\frac{n}{2}-\ell} \int_\alpha^\infty (\lambda-\alpha)^{\frac{n}{2}-\ell} \frac{\mathrm{d}}{\mathrm{d}\lambda}\eta_s(\lambda)\,\mathrm{d}\lambda$$
$$= \frac{1}{\frac{n}{2}-\ell} \int_\alpha^\infty (\lambda-\alpha)^{\frac{n}{2}-\ell}(1+\lambda^2)^{-s}\,\mathrm{d}\lambda.$$

We now use the binomial expansion to obtain

$$\int_\alpha^\infty (\lambda - \alpha)^{\frac{n}{2}-\ell-1} \eta_s(\lambda)\,\mathrm{d}\lambda = \frac{1}{\frac{n}{2}-\ell} \sum_{j=0}^{\frac{n}{2}-\ell} \binom{\frac{n}{2}-\ell}{j} (-\alpha)^{\frac{n}{2}-\ell-j} \int_\alpha^\infty \lambda^j (1+\lambda^2)^{-s}\,\mathrm{d}\lambda.$$

Returning to the residue calculation we find

$$\operatorname*{Res}_{s=\frac{1}{2}} \left(C_s \int_\alpha^\infty (\lambda - \alpha)^{\frac{n}{2}-\ell-1} \eta_s(\lambda)\,\mathrm{d}\lambda \right)$$
$$= \frac{1}{\frac{n}{2}-\ell} \sum_{j=0}^{\frac{n}{2}-\ell} \binom{\frac{n}{2}-\ell}{j} (-\alpha)^{\frac{n}{2}-\ell-j} \operatorname*{Res}_{s=\frac{1}{2}} \left(\int_\alpha^\infty \lambda^j (1+\lambda^2)^{-s}\,\mathrm{d}\lambda \right)$$
$$= \frac{1}{\frac{n}{2}-\ell} \sum_{j=0}^{\frac{n}{2}-\ell} \binom{\frac{n}{2}-\ell}{j} (-\alpha)^{\frac{n}{2}-\ell-j} \operatorname*{Res}_{s=\frac{1}{2}} \left(\int_0^\infty \lambda^j (1+\lambda^2)^{-s}\,\mathrm{d}\lambda \right)$$
$$= \frac{1}{\frac{n}{2}-\ell} \sum_{j=0}^{\frac{n}{2}-\ell} \binom{\frac{n}{2}-\ell}{j} (-\alpha)^{\frac{n}{2}-\ell-j} \operatorname*{Res}_{s=\frac{1}{2}} \left(\frac{\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(s - \frac{j+1}{2}\right)}{2\Gamma(s)} \right)$$
$$= \frac{1}{\frac{n}{2}-\ell} \sum_{\substack{j=0 \\ j \text{ even}}}^{\frac{n}{2}-\ell} \binom{\frac{n}{2}-\ell}{j} \frac{(-1)^{\frac{n}{2}-\ell-\frac{j}{2}} \alpha^{\frac{n}{2}-\ell-j} \Gamma\left(\frac{j+1}{2}\right)}{2\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)},$$

from which the statement follows by observing the relation $c_\ell(n, V) = \left(\frac{n}{2}-\ell\right) C_\ell(n, V)$.
□

4 The spectral flow formula and Levinson's theorem

In this section, we return to the spectral flow formula of Eq. (3.1) applied to the path $H_t(\alpha)$ and, using the results of Sect. 3 we can prove Levinson's theorem in all dimensions.

Theorem 4.1 *Suppose that $V \in C_c^\infty(\mathbb{R}^n)$ and $\alpha > -2\nu$. Then the spectral flow along the path $H_t(\alpha)$ is given by*

$$\mathrm{sf}(H_t(\alpha)) = \frac{1}{2}(N + N_{\mathrm{res}}) + \frac{1}{4\pi i}\int_0^\infty \left(\mathrm{Tr}(S(\lambda)^* S'(\lambda)) - p_n(\lambda)\right) \mathrm{d}\lambda - \frac{1}{2}\beta_n(V).$$

Proof Lemmas 3.1 and 3.6 give that

$$\mathrm{sf}(H_t(\alpha)) = \operatorname*{Res}_{s=\frac{1}{2}}\left(C_s\int_0^1 \mathrm{Tr}\left(V(\mathrm{Id} + H_t(\alpha)^2)^{-s}\right)\mathrm{d}t + \frac{C_s}{2}\mathrm{Tr}\left(\eta_s(H(\alpha)) - \eta_s(H_0(\alpha))\right)\right).$$

Suppose first that n is odd. Applying Proposition 3.4 to the first term on the right-hand side and Proposition 3.8 to the second term gives

$$\mathrm{sf}(H_t(\alpha)) = \frac{1}{2}(N + N_{\mathrm{res}}) + \frac{1}{4\pi i}\int_0^\infty \left(\mathrm{Tr}(S(\lambda)^* S'(\lambda)) - p_n(\lambda)\right) \mathrm{d}\lambda.$$

Now we consider n even. Applying again Propositions 3.4 and 3.8 gives

$$\begin{aligned}\mathrm{sf}(H_t(\alpha)) &= \frac{1}{2}(N + N_{\mathrm{res}}) + \frac{1}{4\pi i}\int_0^\infty \left(\mathrm{Tr}(S(\lambda)^* S'(\lambda)) - p_n(\lambda)\right) \mathrm{d}\lambda \\ &\quad + \frac{1}{2}\sum_{\ell=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\substack{j=0\\ j \text{ even}}}^{\frac{n}{2}-\ell}\binom{\frac{n}{2}-\ell}{j}\frac{(-1)^{\frac{n}{2}-\ell-\frac{j}{2}}\alpha^{\frac{n}{2}-\ell-j}\Gamma\left(\frac{j+1}{2}\right)}{2(2\pi i)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}C_\ell(n,V) \\ &\quad + \sum_{\ell=1}^{\frac{n}{2}}\sum_{\substack{j=0\\ j \text{ even}}}^{\frac{n}{2}-\ell}\binom{\frac{n}{2}-\ell}{j}\frac{(-1)^{\frac{n}{2}-\ell-\frac{j}{2}-1}\alpha^{\frac{n}{2}-\ell-j}\Gamma\left(\frac{j+1}{2}\right)}{4(2\pi i)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}C_\ell(n,V).\end{aligned}$$

It remains to observe that for n even we have $\lfloor\frac{n-1}{2}\rfloor = \frac{n}{2} - 1$ and thus

$$\begin{aligned}&\sum_{\ell=1}^{\frac{n}{2}}\sum_{\substack{j=0\\ j \text{ even}}}^{\frac{n}{2}-\ell}\binom{\frac{n}{2}-\ell}{j}\frac{(-1)^{\frac{n}{2}-\ell-\frac{j}{2}-1}\alpha^{\frac{n}{2}-\ell-j}\Gamma\left(\frac{j+1}{2}\right)}{4(2\pi i)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}C_\ell(n,V) \\ &= -\frac{1}{2}\sum_{\ell=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\substack{j=0\\ j \text{ even}}}^{\frac{n}{2}-\ell}\binom{\frac{n}{2}-\ell}{j}\frac{(-1)^{\frac{n}{2}-\ell-\frac{j}{2}}\alpha^{\frac{n}{2}-\ell-j}\Gamma\left(\frac{j+1}{2}\right)}{2(2\pi i)\Gamma\left(\frac{j}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}C_\ell(n,V) - \frac{1}{2}\beta_n(V),\end{aligned}$$

where we have used the definition of $\beta_n(V)$ in Eq. (2.7). □

We are now able to prove Levinson's theorem as a consequence of spectral flow along the path $H_t(\alpha)$.

Theorem 4.2 (Levinson's theorem) *Suppose that $V \in C_c^\infty(\mathbb{R}^n)$. Then the number N of eigenvalues (counted with multiplicity) of $H = H_0 + V$ is given by*

$$-N = \frac{1}{2\pi i}\int_0^\infty \big(\mathrm{Tr}(S(\lambda)^* S'(\lambda)) - p_n(\lambda)\big)\, \mathrm{d}\lambda - \beta_n(V) + N_{\mathrm{res}},$$

where N_{res} is as defined in Theorem 2.15.

Proof By construction we know that for $\alpha > -2\nu$ we have

$$\mathrm{sf}(H_t(\alpha)) = 0, \tag{4.1}$$

since there is no spectrum which moves through zero from right to left as the path is traversed from $H_0(\alpha)$ to $H(\alpha)$. Substituting Eq. (4.1) into the result of Theorem 4.1 and solving for N completes the proof. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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